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CONTENTS

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STRICT ERGODICITY AND TRANSFORMATION OF THE TORUS.*

By H. Furstenberg.

Introduction. If T is a measure preserving transformation of a probability space Ω with measure μ, the ergodic theorem assures the existence almost everywhere with respect to μ of the average $\lim_{N\to\infty} N^{-1}\sum_{n=0}^{N-1} f(T^n\omega)$, where f is an integrable function. Can this statement be improved in case Ω is a compact topological space, T a suitable homeomorphism of Ω with itself, and f a continuous function on Ω? In particular, can convergence almost everywhere be replaced by convergence everywhere? In what follows we shall examine this question for the case that Ω is an r-dimensional torus. When $r=1$, i.e. Ω is a circle, the answer is in the affirmative and the averages in question always exist (a fact implicit in the results of Denjoy [1] and van Kampen [5]). For $r>1$, however, further restrictions must be imposed on the transformation T and part of our objective will be to exhibit a class of T for which this sharpened form of the ergodic theorem holds.

The question we are considering is closely tied up with that of the "strict ergodicity" of a transformation. A transformation T of a compact Hausdorff space Ω is strictly ergodic if it leaves invariant a unique probability measure on the borel field of Ω. This notion was first introduced in connection with the theory of dynamical systems by Kryloff and Bogoliuboff ([6]; cf. also [8], [9]). When T is a strictly ergodic transformation, then (Theorem 1.1) the limits of $N^{-1}\sum_{n=0}^{N-1} f(T^n\omega)$ necessarily exist for f continuous and all $\omega\in\Omega$, and moreover, this limit is independent of ω. In the case of a strictly ergodic transformation, these conclusions are in fact a good deal more elementary than the usual ergodic theorem. Thus it is quite natural to inquire when a transformation of a given space will be strictly ergodic.

As we will see, an important condition for the validity of some of our conclusions is that the transformation T not be homotopic to the identity transformation. This implies that the transformation T cannot be embedded in a continuous transformation group $T(t)$ and so, in particular, could not arise from the consideration of dynamical systems on the torus. The homotopy

* Received January 19, 1961.

condition also implies that the transformation is not "uniformly almost periodic," and the sample sequences $\xi(n) = f(T^n\omega)$ provide us with new classes of "constructible" sequences possessing mean values without being almost periodic in the sense of Bohr or Besicovitch.

A related problem is that of determining whether the space Ω is "minimal" for the transformation T. A closed invariant set is minimal for T if no proper closed subset is invariant under T. In the situation we shall study, the transformation T will also possess this irreducibility property; that is, Ω will be a minimal set. It will appear that the hypotheses required for minimality and for strict ergodicity are very similar, although not quite identical.

1. Preliminaries.

1.1. Let Ω be a compact metric space, $C(\Omega)$ its algebra of continuous complex valued functions, and T a 1-1 continuous transformation of Ω onto itself. If f is a function on Ω we shall denote by Tf the function satisfying $Tf(\omega) = f(T\omega)$. There always exists some probability meause μ on the borel field of Ω invariant under T. This follows from a general fixed point theorem regarding transformations of a compact convex set. It can also be seen by choosing a countable dense set of functions $\{f_k\}$ in $C(\Omega)$, a point $\omega \in \Omega$, and a sequence N_j such that $\lim_{j\to\infty} N_j^{-1} \sum_{n=0}^{N_j-1} f_k(T^n\omega)$ exists for all k. It follows then that this limit exists for all $f \in C(\Omega)$ and this defines a linear functional L on $C(\Omega)$. L is non-negative for non-negative function, 1 for the function 1, and $L(Tf) = L(f)$. It follows from the Riesz-Markoff representation theorem that L defines an invariant measure.

Definition 1. A continuous transformation T of Ω is called *strictly ergodic* if there is a unique probability measure on the borel field of Ω satisfying $\mu(\Delta) = \mu(T^{-1}\Delta)$ for every borel set Δ in Ω.

We remark that this definition, which is that of ([6]), differs slightly from the more current usage as in [8] and [9]. In the latter, what we have defined is referred to as "unique ergodicity," and a transformation is strictly ergodic only if in addition, the support of the unique invariant measure is all of Ω. Since the support of an invariant measure is a closed invariant set, this means that Ω is required to be a minimal set.

Given a probability measure μ on Ω, then a measure preserving transformation is ergodic in the usual sense if all borel subsets invariant under T have measure either 0 or 1. This is equivalent to saying that no probability measure absolutely continuous with respect to μ is invariant, other

than μ itself. It follows that if T is a strictly ergodic transformation on Ω, it is a fortiori ergodic in the usual sense, relative to the unique invariant measure. The set of all probability measures on Ω invariant under T form a compact convex set, and one may verify that the extremal points of this set are just the ergodic measures for T, i.e. the measures with respect to which T is ergodic. Hence to show that a transformation is strictly ergodic, it suffices to show that there is a unique ergodic measure for the transformation.

We shall refer to the triple (Ω, T, μ) when μ is a probability measure on Ω invariant under T, as a *process*. If we were to fix a function $z(\omega)$ on Ω, then the functions $T^n z(\omega)$ would form a stationary stochastic process. When T is ergodic or strictly ergodic we shall refer to the process as ergodic or strictly ergodic.

Definition 2. A point $\omega \in \Omega$ is a *generic* point for the process (Ω, T, μ), if $\lim N^{-1} \sum_{n=0}^{N-1} f(T^n\omega)$ exists for every $f \in C(\Omega)$, and coincides with $\mu(f) = \int_\Omega f(\omega)\, d\mu(\omega)$.

Note that this definition differs slightly from that given in [2] where it is required in addition that ω lie in the support of the measure μ.

When a process is ergodic it follows from the ergodic theorem that almost every point of Ω is generic. For strictly ergodic processes we have ([9]):

THEOREM 1.1. *The following statements for a process (Ω, T, μ) are equivalent:*

(A) *(Ω, T, μ) is strictly ergodic.*

(B) *$N^{-1} \sum_{n=0}^{N-1} T^n f(\omega)$ converges uniformly to $\mu(f)$.*

(C) *Every point of Ω is generic for (Ω, T, μ).*

Proof. (A) implies (B): If μ is the only probability measure on Ω invariant under T, it follows that the only signed measures invariant under T are $\lambda\mu$, λ real. Hence the uniform closure in $C(\Omega)$ of the subspace $\{Tg - g, g \in C(\Omega)\}$ has codimension 1. Consequently, for any $f \in C(\Omega)$ and $\epsilon > 0$, we can find a $g \in C(\Omega)$ with $|f - \mu(f) - (Tg - g)| < \epsilon$. Hence

$$\left| \sum_0^{N-1} T^n f(\omega) - N\mu(f) - (T^N g - g) \right| < N\epsilon \text{ and } N^{-1} \sum_{n=0}^{N-1} T^n f \to \mu(f)$$

uniformly.

That (B) implies (C) is clear so it remains to show that (C) implies (A).

However, if every point of Ω is generic for (Ω, T, μ), then $N^{-1}\sum_{n=0}^{N-1} T^n f(\omega)$ converges pointwise to $\mu(f)$, and hence $\int N^{-1}\sum_{n=0}^{N-1} T^n f(\omega)\,dv(\omega) \to \mu(f)$ for any probability measure v. In particular if v is invariant under T, then $v(f) = \int N^{-1}\sum_{n=0}^{N-1} T^n f(\omega)\,dv(\omega)$ so that $v(f) = \mu(f)$.

1.2. The best known examples of strictly ergodic processes are those arising from almost periodic transformation.

Definition 3. A transformation T of Ω is *almost periodic* (a.p.) if T leaves no proper ($\neq \emptyset$ or Ω) closed subset of Ω invariant and if the powers T^n form an equicontinuous family of transformation with respect to the metric on Ω. A process (Ω, T, μ), with T almost periodic, will be referred to as an *almost periodic process.*

By the *sample paths* or *sample sequences* of a process (Ω, T, μ), we shall understand the sequences of the form $\xi(n) = T^n f(\omega)$ where $f \in C(\Omega)$. One shows easily that the sample paths of an almost periodic process are (uniformly) almost periodic sequences.

The following result appears in [8]:

THEOREM 1.2. *Every almost periodic transformation is strictly ergodic.*

Proof. The equicontinuity of the T^n implies the equicontinuity of $\{T^n f\}$ for $f \in C(\Omega)$ and hence also the equicontinuity of the averages $\{N^{-1}\sum T^n f\}$. It follows that some subsequence converges uniformly; say the limit function is g. Clearly $Tg = g$ and since T leaves no proper closed subset of Ω invariant it follows that g is a constant. Hence for any probability measure μ, $N^{-1}\sum \mu(T^n f) \to g$ and if μ is invariant, $\mu(f) = g$. Hence T is strictly ergodic.

1.3. To illustrate the general approach to the problem, we begin by examining the case in which Ω is the circle, even though the results here follow from known results regarding transformations of the circle.

THEOREM 1.3. *If T is any* 1-1 *continuous transformation of the circle such that no power of T has a fixed point, then T is strictly ergodic.*

Proof. Let K denote the circle. The hypothesis that no power of T has a fixed point, or equivalently, that T leaves no finite subset of K invariant, implies that any invariant measure must assign measure 0 to individual points, that is, any invariant measure on K is non-atomic. Let ω_0 be a fixed point on K and for two points $\omega_1, \omega_2 \in K$, let $\{\omega_1, \omega_2\}$ denote the open arc

from ω_1 to ω_2 taken counterclockwise. Suppose that μ and μ' are two invariant probability measures on K and let $\lambda = \frac{1}{2}(\mu + \mu')$. We define a map Φ from K to a circle K' by setting $\Phi(\omega) = e^{2\pi i\lambda(\{\omega_0,\omega\})}$. Since λ is non-atomic, Φ will be a continuous map from K onto K'.

We note that for any three points ω_1, ω_2, ω_3 in K,

$$\lambda(\{\omega_1,\omega_2\}) + \lambda(\{\omega_2,\omega_3\}) \equiv \lambda(\{\omega_1,\omega_3\}) \pmod 1.$$

It follows that

$$\begin{aligned}\Phi(T\omega) &= e^{2\pi i\lambda(\{\omega_0,T\omega\})} = e^{2\pi i[\lambda(\{\omega_0,T\omega\})+\lambda(T\{\omega_0,T\omega\})]}\\ &= e^{2\pi i\alpha}e^{2\pi i\lambda(\{T\omega_0,T\omega\})} = e^{2\pi i\alpha}e^{2\pi i\lambda(\{\omega_0,\omega\})} = e^{2\pi i\alpha}\Phi(\omega)\end{aligned}$$

where $\alpha = \lambda(\{\omega_0, T\omega_0\})$. If α were rational we would have $\Phi(T^n\omega) = \Phi(\omega)$ identically for some n. Then $\lambda(\{\omega_0, T^n\omega\} \equiv \lambda(\{\omega_0,\omega\}) \pmod 1$ so that $\lambda(\{\omega, T^n\omega\}) \equiv 0 \pmod 1$ and the interval from ω to $T^n\omega$ carries either measure 0 or 1. Thus either $\lambda(\{\omega, T^n\omega\}) = 0$ or $\lambda(\{T^n\omega, \omega\}) = 0$ and whichever holds for some ω and $T^n\omega$ and it follows that every interval of sufficiently small length on K carries no λ measure. This however is impossible since $\lambda \neq 0$ and this shows that α is irrational.

Thus Φ carries λ into (normalized) lebesgue measure on K' and it takes T into the rotation of K' by an irrational angle. Now μ and μ' are carried by Φ into two measures on K' that are invariant under rotation by an irrational angle, since μ and μ' are invariant under T. But clearly the rotations of a circle preserving a given measure form a closed group. (If $\beta_n \to \beta$ and f is continuous then $f(\theta+\beta_n) \to f(\theta+\beta)$ and $\int f(\theta+\beta_n)dV(\theta) \to \int f(\theta+\beta)dV(\theta)$.) Hence μ and μ' are carried into probability measures on K' invariant under all rotations, i. e. into lebesgue measure. Thus μ and μ' have the same images under Φ. On the other hand, if two intervals of K map onto the same interval of K' under Φ, their symmetric difference carries 0 λ-measure. A fortiori it carries 0 μ- and μ'-measure, and so both intervals have the same μ-measure and μ'-measure. It follows that the measures μ and μ' are determined by their images under Φ and since these agree, $\mu = \mu'$. This proves that T is strictly ergodic.

Corollary. *If T is any continuous transformation of the circle, then* $\lim N^{-1}\sum_{n=0}^{N-1} T^n f(\omega)$ *exists for all continuous functions f and points ω on the circle.*

Proof. If no power of T has a fixed point then the result follows from Theorems 1.1 and 1.3. If some T^m has a fixed point, let $T_1 = T^m$. It is

easily shown then that for every point ω, $T_1{}^n\omega$ converges to a fixed point of T_1. Therefore the averages $\lim N^{-1}\sum_{n=0}^{N-1} T_1{}^n f(\omega)$ exist for all ω. Applying this to the functions $f, Tf, \cdots, T^{m-1}f$ we conclude that $\lim N^{-1}\sum_{n=0}^{N-1} T^n f(\omega)$ exists.

In case T leaves no proper closed subset of K invariant, then the support of the unique invariant measure λ must be all of K. The map Φ of Theorem 1.3 is then a homeomorphism and we can say that the process (K, T, λ) is equivalent to the process induced by an irrational rotation of the circle. In particular the process (K, T, λ) is almost periodic and the sample paths are almost periodic sequences with frequencies $k\alpha$.

2. Processes on the torus.

2.1. The following lemma provides us with an inductive procedure for defining strictly ergodic processes.

LEMMA 2.1. *Let (Ω_0, T_0, μ_0) be a strictly ergodic process, let K be the circle, $K = \{\zeta : |\zeta| = 1\}$, and let m be (normalized) lebesgue measure on K. Take $\Omega = \Omega_0 \times K$ and $T : \Omega \to \Omega$ defined by $T(\omega_0, \zeta) = (T_0\omega_0, g(\omega_0)\zeta)$ where $g(\omega_0)$ is a continuous function with $|g| = 1$. Then T is a strictly ergodic transformation if and only if the equation*

$$g^k(\omega_0) = R(T_0\omega_0)/R(\omega_0) \tag{1}$$

has no solution for k an integer $\neq 0$ and $R(\omega_0)$ a measurable function on (Ω_0, μ_0). In any case, the product measure $\mu_0 \times m$ is an invariant measure for T.

Proof. To verify that $\mu_0 \times m$ is an invariant measure, set $f(\omega) = f_1(\omega_0)f_2(\zeta)$. Then $Tf(\omega) = T_0f_1(\omega_0)f_2(g(\omega_0)\zeta)$ and

$$\begin{aligned}\int_\Omega Tf(\omega)\,d\mu\times m(\omega) &= \int_{\Omega_0}\Big[\int_K f_2(g(\omega_0)\zeta)\,dm(\zeta)\Big]T_0f_1(\omega_0)\,d\mu_0(\omega_0)\\ &= \int_{\Omega_0}\Big[\int_K f_2(\zeta)\,dm(\zeta)\Big]T_0f_1(\omega_0)\,d\mu_0(\omega_0)\\ &= \int_K f_2(\zeta)\,dm(\zeta)\int_{\Omega_0} f_1(\omega_0)\,d\mu_0(\omega_0)\\ &= \int_\Omega f(\omega)\,d\mu\times m(\omega).\end{aligned}$$

We claim that in order that T be strictly ergodic on Ω it suffices that it be ergodic in the ordinary sense with respect to $\mu = \mu_0 \times m$. To see this define

the transformation $S_\beta\colon \Omega\to\Omega$ by $S_\beta(\omega,\zeta)=(\omega_0, e^{2\pi i\beta}\zeta)$. Clearly S_β preserves the measure μ. It follows easily that if ω is a generic point for (Ω, T, μ) then so is $S_\beta(\omega)$ for every $\beta\in[0,1)$. Now suppose that T is ergodic with respect to μ. Then almost all points of Ω (with respect to μ) are generic for (Ω, T, μ). Hence for almost all ω_0 (with respect to μ_0), *all* points (ω_0,ζ) are generic for (Ω, T, μ). Suppose next that there exists some other ergodic measure μ' for T. Since a T-invariant measure on Ω induces a T_0-invariant measure on Ω_0 in a natural way, and since (Ω_0, T_0, μ_0) is strictly ergodic, μ' must induce the same measure μ_0 on Ω_0. Then for almost all ω_0 in Ω_0 there exist points (ω_0,ζ) generic for (Ω, T, μ'). But this is impossible if $\mu'\neq\mu$, since for almost all ω_0, all (ω_0,ζ) are generic for (Ω, T, μ). Thus if $\mu=\mu_0\times m$ is ergodic, it is the unique ergodic invariant measure and hence the unique invariant measure (cf. the remarks following Definition 1).

To determine whether μ is ergodic, suppose that $Tf=f$ where $f\in L^2(\Omega,\mu)$. Since μ is a product measure we can write $f\sim\sum_{-\infty}^{\infty} c_k(\omega_0)\zeta$ with $c_k(\omega_0)\in L^2(\Omega_0,\mu_0)$. $Tf=f$ implies $\sum_{-\infty}^{\infty} c_k(T_0\omega_0)g^k(\omega_0)\zeta^k=\sum_{-\infty}^{\infty} c_k(\omega_0)\zeta^k$ or $c_k(T_0\omega_0)=g^{-k}(\omega_0)c_k(\omega_0)$ for every k. Since T_0 is ergodic, f cannot reduce to a function of ω_0 alone and hence some $c_k(\omega_0)$ is not 0 for $k\neq 0$. Since $|g|=1$ it follows from the ergodicity of T_0 that c_k vanishes only on a set of measure 0, so that $R(\omega_0)=c_k(\omega_0)^{-1}$ is a solution of (1).

Conversely, if (1) has a solution, then the function $R(\omega)\zeta^{-k}$ is invariant under T so that μ is not ergodic. This completes the proof of the lemma.

We note that if (1) has a solution $R(\omega_0)$, then $|R(\omega_0)|$ is almost everywhere constant and we may assume that $|R(\omega_0)|=1$.

2.2. If ϕ is a map from a topological space to the circle K then ϕ is called *essential* if it is not homotopic to the constant map, or equivalently if ϕ does not have a single-valued continuous logarithm. When the space is the circle then the increase in $\frac{1}{2\pi i}\log\phi(\zeta)$ as ζ describes the circle once is an integer referred to as the *degree* of ϕ. Thus ϕ is essential if and only if its degree does not vanish.

Lemma 2.1 provides us with strict ergodicity in case the function $g(\omega_0)$ on Ω_0 satisfies a certain condition with respect to T_0. In the next lemma we find a particular form for (Ω_0, T_0) and $g(\omega_0)$ so that the condition is satisfied. Here Ω_0 is itself the product of a space Ω_1 and the circle, and the points ω_0 of Ω_0 may be represented as (ω_1,ζ), $\omega_1\in\Omega_1$, $\zeta\in K$.

LEMMA 2.2. *Let $\Omega_0=\Omega_1\times K$ where Ω_1 is a connected metric space, let*

μ_1 be a probability measure on Ω_1, Ω, and T_1 a continuous measure preserving transformation of Ω_1. Suppose that T_0: Ω_0 is given by $T_0(\omega_1, \zeta) = (T_1\omega_1, q(\omega_1)\zeta)$ and that T_0 is ergodic with respect to the measure $\mu_1 \times m = \mu_0$ on Ω_0. Let $g(\omega_1, \zeta)$ be a continuous function on Ω_0 such that $g(\omega_1, \zeta)$ is an essential map of $K \to K$ for fixed ω_1 and satisfying a uniform Lipschitz condition in ζ:

$$|g(\omega_1, \zeta') - g(\omega_1, \zeta'')| < M|\zeta' - \zeta''|.$$

Then equation (1), $g^k(\omega_0) = R(T_0\omega_0)/R(\omega_0)$, has no solution for $k \neq 0$ and $R(\omega_0)$ measurable on (Ω_0, μ_0).

Proof. Since Ω_1 is connected, the degree d of the map $g: \omega_1 \times K \to K$ is independent of ω_1, and, in particular, if the map is essential for one ω_1, is is so for all. The degree of g^k will be kd and so g^k is again essential in its second variable and also satisfies a uniform Lipschitz condition. Thus to prove the lemma it suffices to show that $g(\omega_0)$ itself cannot have the form $R(T_0\omega_0)/R(\omega_0)$ with $R(\omega_0)$ a measurable function from Ω to K.

Note that if $d(.,.)$ is a metric for Ω_1, then $d((\omega_1', \zeta'), (\omega_1'', \zeta'')) = \max(d(\omega_1', \omega_1''), |\zeta' - \zeta''|)$ is a metric on Ω_0. Now suppose that $g(\omega_0) = R(T_0\omega_0)/R(\omega_0)$. Since R is a measurable function it is continuous when restricted to some compact subset of Ω_0 with measure $> 1 - \eta$, for arbitrary $\eta > 0$. Hence for a specified δ, $\delta' > 0$, there exists an $\epsilon > 0$ and a subset Λ of Ω with $\mu_0(\Lambda) > 1 - \delta$ and such that $|R(\omega_0') - R(\omega_0'')| < \delta'$ whenever $d(\omega_0', \omega_0'') < \epsilon$ and $\omega_0' \omega_0'' \in \Lambda$. We set

$$\Lambda_1 = \{\omega \in \Omega_1 : m(\zeta : (\omega_1 \zeta) \in \Lambda) \geqq 1 - \delta^{\frac{1}{2}}\}.$$

Then $\Lambda_1 \subset \Omega_1$ and $\mu_1(\Lambda_1) \geqq 1 - \delta^{\frac{1}{2}}$.

If $\delta < 1$, then Λ_1 and hence $\Lambda_1 \times K$ will have positive measure. Therefore there will exist $\epsilon/2$-neighborhoods in Ω_0 intersecting $\Lambda_1 \times K$ in sets of positive measure; let V be such a neighborhood. By the ergodicity of $\mu_0 = \mu_1 \times m$, there will exist points in $V \cap (\Lambda_1 \times K)$ such that infinitely many of their images under T_0^n again lie in $V \cap (\Lambda_1 \times K)$. Let (ω_1', ζ') be such a point and suppose that $T_0^{n_k}(\omega_1', \zeta') \in V \cap (\Lambda_1 \times K)$.

Now define S_β: $\Omega_0 \to \Omega_0$ as in Lemma 2.1; i.e. $S_\beta(\omega_1, \zeta) = (\omega_1, e^{2\pi i\beta}\zeta)$. From the definition of T_0, it is easily checked that each S_β commutes with T_0. Moreover S_β preserves the metric $d(.,.)$ in Ω_0. By applying S_β to (ω_1', ζ') we find that for all ζ, $d(T_0^{n_k}(\omega_1', \zeta), (\omega_1', \zeta)) < \epsilon$, and both (ω_1', ζ) and $T^{n_k}(\omega_1', \zeta)$ lie in $\Lambda_1 \times K$.

From the definition of T_0 we find that $T^{n_k}(\omega_1', \zeta) = (T_1^{n_k}\omega_1', e^{2\pi i\beta_k}\zeta)$ where β_k is independent of ζ. Now since (ω_1', ζ) and $T^{n_k}(\omega_1', \zeta)$ lie in $\Lambda_1 \times K$,

it follows that the set of ζ for which the former is not in Λ has measure $< \delta^{\frac{1}{2}}$, and similarly for the latter. Since the ζ's for the two points are fixed translates of one another it follows that the set of ζ such that either (ω_1', ζ) or $T_0^{n_k}(\omega_1', \zeta)$ is not in Λ has measure $< 2\delta^{\frac{1}{2}}$. Putting all this together we conclude that but for a set of ζ of measure $< 2\delta^{\frac{1}{2}}$,

$$| R(T_0^{n_k}(\omega_1', \zeta)) - \mathrm{R}(\omega_1', \zeta) | < \delta'.$$

To complete the proof of the lemma we observe that $| R(T_0^{n_k}\omega) - R(\omega) | < \delta'$ implies that $| g(T_0^{n_k-1}\omega) \cdots g(T_0\omega) g(\omega) - 1 | < \delta'$. Now we may write $g(T_0^j(\omega_1', \zeta)) = g(T_1^j\omega_1', e^{2\pi i\alpha_j}\zeta) = e^{2\pi i h_j(\theta)}$ where $\zeta = e^{2\pi i\theta}$. Since g satisfies a uniform Lipschitz condition in ζ it follows that the $h_j(\theta)$ satisfy uniformly a Lipschitz condition: $| h_j(\theta_1) - h_j(\theta_2) | < M | \theta_1 - \theta_2 |$. Since g is essential in ζ we will have $h_j(\theta + 1) = h_j(\theta) + d$ where $d \neq 0$ is the degree of g. Set $H_k(\theta) = \sum_{j=0}^{n_k-1} h_j(\theta)$; then $H_k(\theta + 1) = H_k(\theta) + n_k d$ and $| H_k(\theta_1) - H_k(\theta_2) | < M n_k | \theta_1 - \theta_2 |$.

We have then $| e^{2\pi i H_k(\theta)} - 1 | < \delta'$ for all θ outside a set of θ of measure $< 2\delta^{\frac{1}{2}}$. Taking $\delta' \leqq \sqrt{2}$ implies that $H_k(\theta)$ has to remain within a distance $\frac{1}{4}$ from an integer, for all θ outside of a set of measure $< 2\delta^{\frac{1}{2}}$. But $H_k(\theta)$ is a continuous function, and as θ goes from 0 to 1, $H_k(\theta)$ has a range of length at least $n_k d$. This implies that in the exceptional set of measure $< 2\delta^{\frac{1}{2}}$, $H_k(\theta)$ oscillates by $\geqq \frac{1}{2} n_k d$. But by the Lipschitz condition, in a set of measure $< 2\delta^{\frac{1}{2}}$, $H_k(\theta)$ can oscillate by at most $2Mn_k\delta^{\frac{1}{2}}$, and this gives a contradiction if $\delta < (\frac{d}{4M})^2$, which proves the lemma.

Note that the lemma is true, in particular, when Ω_1 reduces to a single point so that Ω_0 is just the circle K.

2.3 THEOREM 2.1. *Let T be a transformation of $\Omega = K^r$ given by*

$$
\begin{aligned}
T\zeta_1 &= e^{2\pi i\alpha}\zeta_1 \\
T\zeta_2 &= g_1(\zeta_1)\zeta_2 \\
&\cdots \\
T\zeta_{j+1} &= g_j(\zeta_1, \cdots, \zeta_j)\zeta_{j+1} \qquad (2)\\
&\cdots \\
T\zeta_r &= g_{r-1}(\zeta_1, \cdots, \zeta_{r-1})\zeta_r
\end{aligned}
$$

where α is irrational, each g_j is a continuous function with $| g | = 1$, each $g_j(\zeta_1, \cdots, \zeta_j)$ satisfies a uniform Lipschitz condition in ζ_j, and $g_j(\zeta_1, \cdots, \zeta_j)$ is of degree d_j in ζ_j where $d_j \neq 0$. Then T is a strictly ergodic transformation and the unique invariant measure is the normalized lebesgue measure on K^r.

Proof. The theorem is valid for $r=1$ by virtue of Theorem 1.2 or Theorem 1.3. Now assume the theorem is valid for $\Omega=K^{r-1}$. By Lemma 2.1 it will follow for $\Omega=K^r$ if the equation

$$g_{r-1}{}^k(\omega_0)=R(T_0\omega_0)/R(\omega_0)$$

has no measurable solution, where T_0 is the induced transformation on K^{r-1} and $\omega_0\in K^{r-1}$. But the space K^{r-1} has the form $K^{r-2}\times K$ and T_0 is induced by T_1 on K^{r-2} in the manner prescribed in Lemma 2.2. Since the hypotheses of Lemma 2.2 are fulfilled, it follows that the equation in question has no solution. The theorem therefore follows by induction on r.

If we denote the invariant probability measure on K^r by μ, we notice that for $i\neq j$, $\mu(\phi(\zeta_i)\psi(\zeta_j))=\mu(\phi(\zeta_i))\mu(\psi(\zeta_j))$, so that for $i\neq j$, ζ_i and ζ_j are independent variables. Taking $j=1$ we obtain in particular

$$\lim N^{-1}\sum_{n=0}^{N-1}T^n\zeta_1{}^l(\omega)T^n\phi(\zeta_i)(\omega)=\mu(\zeta_1{}^l)\mu(\phi(\zeta_i))=0$$

if $l\neq 0$. Since $T^n\zeta_1{}^l=e^{2\pi inl\alpha}\zeta_1{}^l$ this says that the Bohr-Fourier coefficients of the sample sequence $T^n\phi(\zeta_i)(\omega)$ corresponding to the frequencies $l\alpha$ all vanish. Now Theorem 2.1 easily extends to the case where the base process $(T\zeta_1=e^{2\pi i\alpha}\zeta_1)$ is replaced by any almost periodic process. This implies that for any $\phi(\zeta_i)$, $i\neq 1$, the corresponding sample sequences have all their Bohr-Fourier coefficients vanishing except for the constant term. In particular, the sample sequences are not all uniformly almost periodic (or, for that matter, even Besicovitch almost periodic). Hence the transformation T itself cannot be almost periodic. That this is the case could have been deduced from the fact that neither T nor any power of T is homotopic to the identity transformation (since the $d_j\neq 0$). But if T is almost periodic then some sequence, T^{n_j} converges uniformly to the identity, and since the homotopy classes of maps of $K^r\to K^r$ form a discrete set, this cannot happen unless eventually T^{n_j} is homotopic to the identity transformation.

Remark. Theorem 2.1 implies in particular that the transformations (2) leave no proper closed subset of K^r invariant. It should be pointed out that for this weaker conclusion one can dispense with the Lipschitz condition on the g; (but not with the condition that the $d_j\neq 0$). In fact one can prove an analogue of Lemma 2.1 with strict ergodicity replaced by the condition that no proper closed sets are invariant. Here the condition on $g(\omega_0)$ becomes that equation (1) has no solution with $R(\omega_0)$ continuous, rather than measurable. However, this is the case if $g(\zeta_1,\cdots,\zeta_i)$ is essential as a function of ζ_j. For $R(\zeta_1,\cdots,\zeta_j)$ would have a certain degree in ζ_j and one

verifies from (2) that $TR(\zeta_1,\cdots,\zeta_j)$ has the same degree in ζ_j. Then $TR(\zeta_1,\cdots,\zeta_j)/R(\zeta_1,\cdots,\zeta_j)$ would have degree 0 in ζ_j and could not equal $g(\zeta_1,\cdots,\zeta_j)$ which is essential in ζ_j. Thus the smoothness condition on the g_j's in Theorem 2.1 is needed only to ensure that no measurable solution exist to (1).

2.4. The sample sequence of a strictly ergodic process not only possess averages but these averages are taken on with a certain uniformity. More precisely

$$N^{-1}\sum_{k+1}^{k+N} T^n f(\omega) \to \mu(f)$$

uniformly in k as $N \to \infty$. This is an immediate consequence of Theorem 1.1. This notion occurs in [10] and [7] in the definition of a "well-distributed" sequence. A sequence $\zeta(n)$ on the unit circle is well-distributed if $\lim_{N\to\infty} N^{-1}\sum_{k+1}^{k+N} f(\zeta(n)) = \int_k f(\zeta)\,dm(\zeta)$ uniformly in k, thus strengthening the condition for equidistribution. We might observe that the theorem ([7]) that if $p(n)$ is a polynomial in n with an irrational coefficient, then $e^{2\pi i p(n)}$ is well distributed, is a consequence of Theorem 2.1. In fact, by an appropriate choice of integers ν_{ij} and a point $\omega \in K^r$, the sample sequence $T^n\zeta_r(\omega)$ will have the form $e^{2\pi i(\alpha n^r+\alpha_1 n^{r-1}+\cdots+\alpha_r)}$ if the functions g_j have the form $g_j(\zeta_1,\cdots,\zeta_j) = \zeta_1^{\nu_{j1}}\cdots\zeta_j^{\nu_{jj}}$. The "well-distribution" of $T^n\zeta_r(\omega)$ then follows from the fact that the unique stationary measure on K^r is lebesgue measure. This takes care of the case that $p(n)$ has its first coefficient irrational; the general case easily reduces to this one.

3. Existence of ergodic averages.

3.1. It is possible to show by examples that the conditions on the smoothness and the topological nature (i. e., the non-vanishing of degree) of the functions g_j occurring in Theorem 2.1 cannot be essentially weakened. For example, if the Lipschitz condition is replaced by the requirement that the g_j be of bounded variation, then the conclusion of Theorem 2.1 fails to hold, as a rather intricate construction shows. The necessity of the topological assumption is more evident since, e. g., the map: $T\zeta_1 = e^{2\pi i\alpha}\zeta_1$, $T\zeta_2 = e^{2\pi i\rho\alpha}\zeta_2$ of the torus K^2 is clearly not strictly ergodic if ρ is rational. In the latter case, however, the ergodic averages, $\lim N^{-1}\sum_{n=0}^{N-1} T^n f(\omega)$ still exist for all ω, although they are no longer independent of ω. We will see in this section that the hypotheses of Theorem 2.1 are still relevant for the weaker conclusion of existence everywhere of the ergodic averages.

THEOREM 3.1. *Let* (Ω_0, T_0, μ_0) *be a strictly ergodic process, let* $\Omega = \Omega_0 \times K$ *and let* $T: \Omega \to \Omega$ *be defined by* $T(\omega_0, \zeta) = (T_0\omega_0, g(\omega_0)\zeta)$ *where* g *is a continuous function from* Ω_0 *to* K. *Then if the equation*

$$g^k(\omega_0) = R(T_0\omega_0)/R(\omega_0) \tag{1'}$$

has a solution $R(\omega_0)$ *which is measurable but not equal almost everywhere to a continuous functions, then* $\lim N^{-1} \sum_{n=0}^{N-1} T^n f(\omega)$ *cannot exist for all continuous* f *and all* $\omega \in \Omega$.

Proof. Let $\phi(\omega)$ be a continuous function of Ω_0 with $\mu_0(\phi \cdot R) \neq 0$, and consider the expressions $T^n(\phi(\omega_0)\zeta^k)$. We have

$$T\zeta^k = g^k(\omega_0)\zeta^k = R(T_0\omega_0)R(\omega_0)^{-1}\zeta^k, \text{ or } T(\zeta^k R(\omega_0)^{-1}) = \zeta^k R(\omega_0)^{-1},$$

whence $T^n(\zeta^k R(\omega_0)^{-1}) = \zeta^k R(\omega_0)^{-1}$, so that

$$T^n(\phi(\omega_0)\zeta^k) = \zeta^k R(\omega_0)^{-1} T_0{}^n(\phi(\omega_0)R(\omega_0)). \tag{3}$$

Suppose now that all ergdoic everages exist for continuous functions on Ω. Then the average of the left hand side of (3) exists, and so

$$\zeta^k R(\omega_0)^{-1} \lim N^{-1} \sum_{n=0}^{N-1} T^n(\phi(\omega_0)R(\omega_0))$$
$$= \lim N^{-1} \sum_{n=0}^{N-1} T^n(\phi(\omega_0)\zeta^k) = \psi(\omega_0, \zeta)$$

is defined. Since (Ω_0, T_0, μ_0) is ergodic we have

$$\zeta^k R(\omega_0)^{-1}\mu_0(\phi \cdot R) = \psi(\omega_0, \zeta)$$

almost everywhere. Since $\mu_0(\phi \cdot R) \neq 0$ and $\psi(\omega_0, \zeta)$ is a limit everywhere of continuous functions on Ω, it follows that $R(\omega_0)$ agrees almost everywhere with a function $\bar{R}(\omega_0)$ of Baire class 1 on Ω, and hence on Ω_0. We shall show that this together with (1′) implies that $R(\omega_0)$ agrees almost everywhere with a continuous function, contrary to hypothesis.

We say that a function $f(\omega_0)$ has an essential jump $\geqq \delta$ at a point ω_0, if, for any function $\bar{f}$ equal to f a.e., $\limsup_{\omega', \omega'' \to \omega_0} |\bar{f}(\omega') - \bar{f}(\omega'')| \geqq \delta$. Thus if R is not equivalent to a continuous function, then R, and hence $\bar{R}$, must have essential jumps $\geqq \delta$ for some $\delta > 0$. Now equation (1′) still holds a.e. for $\bar{R}$ and from it we see that the set of points at which $\bar{R}$ has an essential jump $\geqq \delta$ is invariant under T_0. Moreover the set of points at which a function has an essential jump $\geqq \delta$ is closed. Now if (Ω_0, T_0, μ) is strictly ergodic, every invariant closed subset of Ω must contain the support of μ_0. Since we may assume that the support of μ_0 was all of Ω_0 it follows that $\bar{R}(\omega_0)$ has

essential jumps everywhere. But by Baire's theorem, a function of the first class must have points of continuity ([4]). This proves the theorem.

If the space Ω_0 in the theorem is itself of the form $\Omega_0 = \Omega_1 \times K$ and T_0 has a form analogous to that of T, and if $g(\omega_0) = g(\omega_1, \zeta)$ is essential in ζ, then equation (1′) cannot have a continuous solution (cf. the remark in § 2.3). Thus, if in this case the ergodic averages exist everywhere, then T must be strictly ergodic. So, in the example referred to earlier, where the g_j in Theorem 2.1 are of bounded variation, but do not satisfy a Lipschitz condition, and the process need not be strictly ergodic, neither will the ergodic averages all exist.

Using Theorem 3.1 we can give an example of a map T of K^2 of the form (2), where T is a C^∞ transformation, but fails to satisfy the topological hypothesis, and such that the ergodic averages do not all exist. Define a sequence of integers v_k with $v_1 = 1$ and $v_{k+1} = 2^{v_k} + v_k + 1$. Set $\alpha = \sum_{\infty}^{1} 2^{-v_k}$. Writing $n_k = 2^{v_k}$ we obtain

$$(3) \qquad | n_k\alpha - [n_k\alpha] | > \frac{2 \cdot 2^{v_k}}{2^{v_{k+1}}} = 2^{-n_k}$$

where $[x]$ denotes the largest integer not exceeding x.

Define $n_{-k} = -n_k$ and set

$$h(\theta) = \sum_{k \neq 0} \frac{1}{|k|} (e^{2\pi i n_k \alpha} - 1) e^{2\pi i n_k \theta}$$

and let $g(e^{2\pi i \theta}) = e^{2\pi i \lambda h(\theta)}$ where λ is as yet undetermined. Note that $g(\zeta)$ is inessential. By (3), $h(\theta)$ and hence $g(\zeta)$ are C^∞ functions. Now we have $h(\theta) = H(\theta + \alpha) - H(\theta)$ where

$$H(\theta) = \sum_{k \neq 0} \frac{1}{|k|} e^{2\pi i n_k \theta}$$

so that $H(\theta)$ is in $L^2(0,1)$ and, in particular, defines a measurable function. However $H(\theta)$ cannot correspond to a continuous function since $\sum \frac{1}{|k|} = \infty$ and hence the series is not Cesàro summable at $\theta = 0$ ([11]). Therefore $e^{2\pi i \lambda H(\theta)}$ will not be a continuous function for some λ. Taking $R(e^{2\pi i \theta}) = e^{2\pi i \lambda H(\theta)}$ we have

$$R(e^{2\pi i \alpha}\zeta)/R(\zeta) = g(\zeta)$$

with $R(\zeta)$ measurable but not continuous. By Theorem 3.1, the transformation T of K^2 given by

$$T\zeta_1 = e^{2\pi i \alpha}\zeta_1, \qquad T\zeta_2 = g(\zeta_1)\zeta_2,$$

will not possess all its ergodic averages.

3.2. The transformations given by (2) are in a "triangular" form, and this, we may note, implies certain restrictions in the homotopy classes of the transformations under consideration. Fixing our attention on the case $r=2$, we note that the induced transformation on the 1-dimensional integral homology group is given by a matrix

$$\begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$$

where d is the degree of $g(\zeta_1)$. Namely, the homology group is generated by the classes of the cycles $\rho_1=\{\zeta_2=1\}$, $\rho_2=\{\zeta_1=1\}$. Under the transformation

$$T\zeta_1 = e^{2\pi i}\zeta_1 \qquad (4)$$
$$T\zeta_2 = g(\zeta_1)\zeta_2$$

ρ_1 is transformed into $\rho_1+d\rho_2$ and ρ_2 into ρ_2 in the homology group. Now the homotopy classes of homeomorphisms of K^2 with itself are in one-one correspondence with the induced transformation of the first integral homology group, and so may be represented by 2×2 unimodular matrices with integer coefficients. Thus the homotopy classes that are represented by the transformations in (4) are just those whose matrices have both eigenvalues 1, or equivalently, have trace 2. Now this restriction is quite natural if we are interested in strictly ergodic transformations, since if the associated matrix does not have trace 2, the map necessarily has a fixed point. This can be shown directly as a consequence of the fact that any two cycles on K^2 that are not homologous to multiples of the same cycle, must intersect. It is also a simple corollary of the Lefschetz fixed point theorem. However, a fixed point of the transformation is clearly not generic for a process unless the process degenerates to that one point, and so in any other case, the transformation cannot be strictly ergodic.

The next theorem indicates that this restriction (with a slight modification) is also necessary for the existence of ergodic averages. We call a transformation of the torus linear if it is induced by a linear transformation of the plane. In order for this to happen, the associated matrix of the linear transformation on the plane must be a unimodular matrix with integer entries. It is easy to verify that this same matrix represents the homology class of the induced transformation of the torus.

THEOREM 3.2. *Let T be a transformation of the torus induced by the transformation*

$$x' = ax + by + x'_0 \qquad (5)$$
$$y' = cx + dy + y'_0$$

of the plane R^2, where a, b, c, d are integers and $ad - bc = 1$. Then $\lim N^{-1} \sum_{n=0}^{N-1} T^n f(\omega)$ exists for every continuous function f on K^2 and $\omega \in K^2$, if and only if $|a + d| \leqq 2$.

Proof. If $a + d = 2$ then both eigenvalues of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are 1 in which case, by a linear change of coordinates, (5) may be put into the form

$$(6) \qquad \begin{aligned} x' &= x + x'_0 \\ y' &= dx + y + y'_0. \end{aligned}$$

If $d = 0$, or if $d \neq 0$ and x'_0 is rational, it is easily checked that the sample sequences of the induced process are periodic or almost periodic. If $d \neq 0$ and x'_0 is irrational then the ergodic averages exist by virtue of Theorem 2.1.

If $-2 \leqq a + d < 2$, then the eigenvalues of A are roots of unity and an appropriate power of A will have both eigenvalues 1. Hence a power of T will fall into the category already treated, and the existence of ergodic averages for a power of T implies the existence of these averages for T itself.

We may therefore turn to the case $|a + d| > 2$. Here we wish to show that the ergodic averages need not exist. Since in this case the induced transformation of the torus has a fixed point, we may assume that the fixed point corresponds to the origin in the plane and that T in the plane is given by

$$(7) \qquad \begin{aligned} x' &= ax + by \\ y' &= cx + dy, \quad |a + d| > 2. \end{aligned}$$

We now consider T solely as a transformation of R^2. We denote by Z^2 the lattice of points of R^2 with integer coordinates, and for a set Δ in R^2, $\Delta + Z^2$ will denote the set of all points in R^2 congruent to a point in Δ modulo Z^2. We let l_1 and l_2 represent the lines through the origin containing the eigenvectors of A; l_1 will correspond to the eigenvalue λ, and l_2 to the eigenvalue λ^{-1}. Since $|\lambda + \lambda^{-1}| > 2$, λ and λ^{-1} will be real and we take $|\lambda| > 1$. Also λ is not an integer since $\lambda + \lambda^{-1}$ is, and since $\lambda^2 - (a + d)\lambda + 1 = 0$, it follows that λ is irrational. Hence the slopes of l_1 and l_2 are irrational and so the sets $l_1 + Z^2$ and $l_2 + Z^2$ are dense in R.

We consider parallelograms whose sides are respectively parallel to l_1 and l_2; we call these oriented parallelograms. The "width" of the parallelogram will denote the length of the side parallel to l_1, its "height," the length of the adjacent side. T takes one oriented parallelogram into another one, multiplying its width by $|\lambda|$ and its height by $|\lambda|^{-1}$. Let Q_1 and Q_2 be two oriented rhombuses with centers of the origin of R^2, and with sides equal to ϵ

and 2ϵ respectively. ϵ will be determined later. To prove that not all ergodic averages exist for the induced transformation of the torus we will show that there exists a point ω in R^2 with

$$\limsup_{N\to\infty} N^{-1}\{\#\text{ of points } T^n\omega,\ 0 \leqq n < N, \text{ such that } T^n\omega \in Q_1 + Z^2\}$$
$$> \liminf_{N\to\infty} N^{-1}\{\#\text{ of points } T^n\omega,\ 0 \leqq n < N, \text{ such that } T^n\omega \in Q_2 + Z^2\}.$$

Since $Q_1 \subset Q_2$ this implies that for a doubly periodic continuous function f on R^2 that equals 1 on Q_1 and vanishes outside of Q_2, $\lim N^{-1}\sum T^n f(\omega)$ does not exist. But this implies that for the induced transformation on K^2 and the associated function on K^2, the ergodic average doesn't exist at all points.

To obtain the point ω we prove the following two statements.

(a) Given an oriented parallelogram P there exists an N_0 such that for any $N > N_0$ there is an oriented parallelogram $P_a \subset P$ with the property that for any $\omega \in P_a$, the number of times $T^n\omega \in Q_2 + Z^2$ for $0 \leqq n < N$, is $\leqq \beta_1\epsilon^2 N$, β_1 being a constant depending only on A.

(b) Given a noriented parallelogram P there exists an N_0 such that for any $N > N_0$ there is an oriented parallelogram $P_b \subset P$ with the property that for any $\omega \in P_b$ the number of times $T^n\omega \in Q_1 + Z^2$ for $0 \leqq n < N$, is $\geqq (\beta_2 + \beta_3 \log \epsilon^{-1})^{-1}N$ where β_2 and β_3 depend only on A.

The proof of (a) depends on the fact that the transformation T on the torus is ergodic with respect to lebesgue measure (as a straightforward Fourier series argument shows). Hence for almost all $\omega \in R^2$, the frequency of entering $Q_2 + Z^2$ is the area of Q_2, which is proportional to ϵ^2, the constant of proportionality depending only on A. Hence for some ω in P, this will be true and the inequality in (a) holds for this point for every sufficiently large N. But if it is true for ω and a fixed N, it is also true for some neighborhood P_a of ω for that N.

For the proof of (b), let $l_2 + v_0$ be a translate of l_2 by a vector v_0 in Z^2 which intersects P. For some positive integer m_0, the parallelogram $T^{-m_0}Q_1 + v_0$ will intersect P, thereby dividing P into 3 parallelograms, since $T^m Q_1$ tends to l_2 as $m \to \infty$. Let P_1 denote $P \cap (T^{-m_0}Q_1 + v_0)$. Then $T^{m_0}(P_1) \subset Q_1 + Z^2$. Consider an oriented parallelogram with center at v_0 whose width is $\frac{1}{2}$ the width of $T^{-m_0}Q_1 + v_0$ and whose length is the length of $T^{-m_0}Q_1 + v$ multiplied by $8\gamma^{-1}\epsilon^{-2}$ where γ is the sine of the angle between l_1 and l_2. $\gamma\epsilon^2$ is the area of Q_1, and thus the area of the resulting paralelogram is 4. It follows by the Minkowski theorem on convex bodies, that this parallelogram contains a lattice point v_0. Then it can be seen that $T^{-m_1}Q_1 + v_1$

will intersect P_1, dividing it into 3 parts, provided that $|\lambda|^{m_1-m_0} \geqq 16\gamma^{-1}\epsilon^{-2}$. We may therefore take

$$m_1 = m_0 + [(\log|\lambda|)^{-1}\epsilon^{-2}] + 1 \leqq m_0 + \beta_2' + \beta_3' \log \epsilon^{-1}$$

where β_2' and β_3' depend only on A. We denote the intersection of $T^{-m_1}Q_1 + v_1$ and P_1 by P_2. Then $T^{m_0}(P_2) \subset Q_1 + Z^2$, $T^{m_1}(P_2) \subset Q_1 + Z^2$. In a similar manner we can find a lattice point v_2 in a parallelogram related to $T^{-m_1}Q_1 + v_1$ in the same way that the foregoing parallelogram of area 4 was related to $T^{-m_0}Q_1 + v_0$. For this v_2 we will have $T^{-m_2}Q_1 + v_2$ cutting P_2 into 3 portions, where $m_2 \leqq m_1 + \beta_2' + \beta_3 \log \epsilon^{-1}$. The intersection P_3 will satisfy $T^{m_i}(P_3) \subset Q_1 + Z^2$ for $i = 0, 1, 2$. Continuing in this way, we obtain parallelograms P_n with $T^{m_i}(P_n) \subset Q_1 + Z^2$ for $i = 0, 1, \cdots, n-1$, and with the integers $m_0, m_1, \cdots, m_{n-1}$ separated by a distance not exceeding $\beta_2' + \beta_3' \log \epsilon^{-1}$. Now take $\beta_2 = 2\beta_2'$, $\beta_3 = 2\beta_3'$, and $N = 2m$. The parallelogram P_b is then chosen as the P_n for $m_{n-1} < N \leqq m_n$. For then, the number of times T^kP_b enter $Q_1 + Z^2$ exceeds n, and since $N \leqq m_n < m_0 + n(\beta_2' + \beta_3' \log \epsilon^{-1}) < N/2 + n/2(\beta_2 + \beta_3 \log \epsilon^{-1})$, we have

$$n > (\beta_2 + \beta_3 \log \epsilon^{-1})^{-1}N.$$

Now suppose ϵ to have been chosen so small that

$$\beta_1\epsilon^2 < (\beta_2 + \beta_3 \log \epsilon^{-1})^{-1}.$$

Let P be an arbitrary oriented parallelogram in the plane and form the parallelograms P_a, $P_{ab} = (P_a)_b$, $P_{abc} = (P_{ab})_c$, etc., for a sequence of $N \to \infty$. Take ω as any point in the intersection of all the $P_{ab\cdots ab}$. Since ω belongs to infinitely many $P_{ab\cdots ab}$,

$$\limsup_{N\to\infty} N^{-1}\{\#\text{ of } T^n\omega,\ 0 \leqq n < N, \text{ such that } T^n\omega \in Q_1 + Z^2\} \geqq (\beta_2 + \beta_3 \log \epsilon^{-1})^{-1}.$$

On the other hand since ω belongs to infinitely many $P_{ab\cdots ba}$,

$$\liminf_{N\to\infty} N^{-1}\{\#\text{ of } T^n\omega,\ 0 \leqq n < N, \text{ with } T^n\omega \in Q_2 + Z^2\} \leqq \beta_1\epsilon^2.$$

This ω has the desired properties and this proves the theorem.

COROLLARY. *If* λ *satisfies* $\lambda^2 - a\lambda + 1 = 0$ *where* a *is an integer, and* $|\lambda| > 1$, *then there exists a real number* t *such that* $\lim N^{-1} \sum_{n=0}^{N-1} e^{2\pi i\lambda^n t}$ *does not exist.*

Proof. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$; A has eigenvalues λ and λ^{-1}. Let e_1 and e_2

be two associated eigenvectors in R^2, $e_1=\begin{pmatrix} e_1' \\ e_1'' \end{pmatrix}$, $e_2=\begin{pmatrix} e_2' \\ e_2'' \end{pmatrix}$. Now let $u=xe_1+ye_2$ be any vector in R^2. $T^n u=\lambda^n xe_1+\lambda^{-n}ye_2$. The ergodic averages of $f(T^n u)$ will exist for any continuous doubly periodic function f if they exist for $f(u)=e^{2\pi i(ku'+lu'')}$ where $u=\begin{pmatrix} u' \\ u'' \end{pmatrix}$. Since $\lambda^{-n}\to 0$ it follows that the ergodic averages exist for all f if $\lim N^{-1}\sum_{n=0}^{N-1} e^{2\pi i\lambda^n(kxe_1'+lxe_1'')}$ exists for k, l, and x. By the foregoing theorem this is not the case and hence with $t=kxe_1'+lxe_1''$ for and appropriate choice of k, l, x, $\lim N^{-1}\sum e^{2\pi i\lambda^n t}$ will not exist.

In fact, the conclusion of this corollary requires only the hypothesis that $|\lambda|>1$. This was pointed out to the author by Helson who has a very elegant proof of this result.

4. Extensions of strictly ergodic processes.

4.1. In § 2 we studied processes that were defined inductively from lower dimensional processes in the following manner: To each point ω_0 of a process (Ω_0, T_0, μ) we associate the circle $\omega_0\times K$, and Ω is taken as the union of these circles. A tranformation T is defined on Ω by allowing T to permute the circles $\omega_0\times K$ among themselves in accordance with T_0, and each circle is rotated by an amount depending on ω_0. In this section, we consider the more general situation where the transformation T on the individual circles is not necessarily a rotation, but any homeomorphism of the circle.

Definition 4. A process (Ω, T, μ) *extends* a process (Ω_0, T_0, μ_0) (and (Ω_0, T_0, μ_0) is a *subprocess* of (Ω, T, μ)), if there is a continuous map Φ taking Ω onto Ω_0 with $\Phi T=T_0\Phi$ and with $\mu_0=\Phi(\mu)$ in the sense that for any borel set Δ in Ω_0, $\mu_0(\Delta)=\mu(\Phi^{-1}(\Delta))$.

We notice that any subprocess of (Ω, T, μ) determines a subalgebra of $C(\Omega)$ by taking the set of functions $f=g\circ\Phi$. This subalgebra is isomorphic to $C(\Omega_0)$ and T_0 on $C(\Omega_0)$ is induced by T on $C(\Omega)$. Conversely one may show that any closed subalgebra of $C(\Omega)$, invariant under T, determines a subprocess. For example, the process $(K^r, T^{(r)}, m^r)$ defined by (2) in § 2 is an extension of $(K^s, T^{(s)}, m^s)$ for $s<r$ defined by restricting (2) to the first s coordinates $(\zeta_1,\cdots,\zeta_s)$. The subprocess thus corresponds to the subalgebras of functions on K^r depending only on $(\zeta_1,\cdots,\zeta_s)$. This subalgebra is invariant under T because each $T\zeta_i$ is a function of $\zeta_1,\cdots,\zeta_i$.

Definition 5. An extension (Ω, T, μ) of (Ω_0, T_0, μ_0) is *finite* if there

is an integer $h > 0$ and a subset Λ of Ω of measure 0, such that almost every point $\omega_0 \in \Omega_0$ has at most h preimages in $\Omega - \Lambda$.

If $h = 1$, then (Ω, T, μ) is what has been called in [2] and L-extension of (Ω_0, T_0, μ_0). This means that the continuous functions on the extension Ω may be identified with measurable functions on Ω_0. Conversely under these circumstances h can be taken as 1.

The transformation T of K^2 given by $T\zeta_1 = e^{2\pi i\alpha}\zeta_1$, $T\zeta_2 = e^{2\pi i\alpha/h}\zeta_2$ leaves invariant a cycle on K^2. Consequently T defines a finite extension of (K, S_α, m) where $S_\alpha\zeta = e^{2\pi i\alpha}\zeta$.

The following theorem can be regarded as a generalization of Theorem 1. 3.

THEOREM 4. 1. *Let (Ω_0, T_0, μ_0) be a strictly ergodic process, let $\Omega = \Omega_0 \times K$, and let T be any continuous transformaton of Ω of the form $T(\omega_0, \zeta) = (T_0\omega_0, g(\omega_0, \zeta))$ where $g: \Omega \to K$. Then either T is a strictly ergodic transformation of Ω or there exists an invariant measure μ such that (Ω, T, μ) is a finite extension of (Ω_0, T_0, μ_0). Moreover, in the latter case, every invariant measure μ defines a finite extension.*

Remarks. If we define $\Phi: \Omega \to \Omega_0$ by $\Phi(\omega_0, \zeta) = \omega_0$, then the condition on T is just that T should extend T_0, i. e., that $\Phi T = T_0\Phi$. Since $\Phi(\mu)$ will therefore be invariant under T_0 if μ is invariant under T, and since μ_0 is the unique invariant measure for T_0, it follows that under these conditions (Ω, T, μ) necessarily extends (Ω_0, T_0, μ_0). We remark that Theorem 1. 3 is obtained from this theorem by taking for (Ω_0, T_0, μ_0) the trivial process for which Ω_0 consists of a single point.

Proof. Let M be the compact space of probability measures on K. An element of M is determined by a positive functional on $C(K)$ taking the value 1 at 1. We shall need the following observation. If H is a dense subset of $R(K)$, the set of real valued continuous functions on K, such that H is closed under additions, multiplication by rationals, and the lattice operations $\min(f, g)$, $\max(f, g)$, and $f \equiv 1 \in H$, then any functional on H which is linear over the rationals, which takes non-negative values for non-negative functions in H, and which takes the value 1 for the function 1 determines a unique measure in M. The reason is that any such functional can be extended to a linear, positive functional on $C(K)$, as one may easily show. The significance of this is due to the fact that H may be taken to be an enumerable subset of $C(K)$, and thus an enumerable number of conditions are imposed on a functional in order that it correspond to a measure in M.

Now consider the subsets of $\Omega = \Omega_0 \times K$ of the form $\Delta \times K$ with Δ a

borel subset of Ω_0. These form a subfield B_0 of the borel field of Ω, and so we may speak of conditional expectations with respect to B_0. By the foregoing remarks it will follow that there exists an M-valued function on Ω_0, $\mu(\omega_0)$, such that these conditional expectations are obtained by

$$E(f|B_0)(\omega_0)=\int_K f(\omega_0,\zeta)\,d\mu(\omega_0)(\zeta) \tag{8}$$

for any bounded borel measurable function f on Ω. To construct $\mu(\omega_0)$ we apply (8) to functions $f(\omega_0,\zeta)=g(\zeta)$ where $g\in H$. The left hand side is defined almost everywhere and this defines a functional on H a. e. The fact that the functional may be defined on all of H a. e., is a consequence of the countability of H. Moreover from (8) if follows that at almost all ω_0, the functional obeys the enumerable set of conditions imposed in the preceding paragraph. Thus $\mu(\omega_0)$ is defined and it may be seen to be a measurable function from Ω_0 to M with the borel field on M induced by the weak topology on M (i. e. the topology for which M is compact: $\mu_n\to\mu$ if $\mu_n(f)\to\mu(f)$ for $f\in C(K)$). It follows that (8) is valid for f depending only on ζ, and from this, (8) follows for $f(\omega_0,\zeta)$ of the form $f_1(\omega_0)f_2(\zeta)$. It is clear then that (8) hold a. e. for any bounded borel measurable function on Ω.

From (8) and $E(E(f|B_0))=E(f)$ we obtain

$$\int_{\Omega_0}\{\int_K f(\omega_0,\zeta)\,d\mu(\omega_0)(\zeta)\}d\mu(\omega)=\int_\Omega f(\omega)\,d\mu(\omega). \tag{9}$$

We note also that (9) in turn uniquely determines the measurable function $\mu(\omega_0)$ up to sets of μ_0 measure 0, as can be seen by taking $f(\omega_0,\zeta)=f_1(\omega_0)f_2(\zeta)$.

Let $\{\zeta_1,\zeta_2\}$ denote, as in Theorem 1.3, the open arc on K from ζ_1 to ζ_2, taken counterclockwise. Let us define an M-valued function $\mu'(\omega_0)$ by

$$\mu'(\omega_0)(\{\zeta_1,\zeta_2\})=\mu(T\omega_0)(\{g(\omega_0,\zeta_1),g(\omega_0,\zeta_2)\})$$

where $T(\omega_0,\zeta)=(T_0\omega_0,g(\omega_0,\zeta))$. We then have

$$\begin{aligned}
&\int_{\Omega_0}\{\int_K f(\omega_0,\zeta)\,d\mu'(\omega_0)(\zeta)\}d\mu_0(\omega_0)\\
&=\int_{\Omega_0}T_0\{\int_K f(T^{-1}(\omega_0,\zeta))\,d\mu(\omega_0)(\zeta)\}d\mu_0(\omega_0)\\
&=\int_{\Omega_0}\{\int_K f(T^{-1}(\omega_0,\zeta))\,d\mu(\omega_0)(\zeta)\}d\mu_0(\omega_0)=\int_\Omega f(T^{-1}\omega)\,d\mu(\omega)\\
&=\int_\Omega\{\int_{\Omega_0} f(\omega)\,d\mu(\omega)=\int_K f(\omega_0,\zeta)\,d\mu(\omega_0)(\zeta)\}d\mu_0(\omega_0).
\end{aligned}$$

But then $\mu'(\omega_0)=\mu(\omega_0)$ a. e. so that we have:

$$\mu(\omega_0)(\{\zeta_1,\zeta_2\})=\mu(T_0\omega_0)(\{g(\omega_0,\zeta_1),g(\omega_0,\zeta_2)\}) \tag{10}$$

almost everywhere. This says that T carries the associated measure $\mu(\omega_0)$ on $\omega_0 \times K$ onto the associated measure $\mu(T_0\omega_0)$ on $T_0\omega_0 \times K$. We shall refer to the $\mu(\omega_0)$ as the *relative* measures.

The condition that (Ω, T, μ) be a finite extension of (Ω_0, T_0, μ_0) may now be restated as the condition that for some h, the relative measures, $\mu(\omega_0)$, have their support, for almost all ω_0, on finite subsets of K having no more than h points. If, on the one hand, (Ω, T, μ) is a finite extension of (Ω_0, T_0, μ_0), then the measure $\mu'(\omega_0)$ obtained by restricting $\mu(\omega_0)$ to the points of K corresponding to $(\omega_0 \times K) \cap (\Omega - \Lambda)$, must agree a.e. with $\mu(\omega_0)$ since $\mu(\Lambda) = 0$. Hence $\mu(\omega_0)$ has its support a.e. on a set of $\leqq h$ points. Conversely suppose that $\mu(\omega_0)$ has its support a.e. on $\leqq h$ points. Define Λ as the union of the points (ω_0, ζ) for which ζ is not in the support of $\mu(\omega_0)$. We claim that Λ is a measurable set in Ω. To see this, take Γ_ϵ a compact set in Ω_0, with $\mu_0(\Gamma_\epsilon) > 1 - \epsilon$, $\epsilon > 0$, such that $\mu(\omega_0)$ is a continuous function from Γ_ϵ to M. This is possible because of the measurability of $\mu(\omega_0)$. On Γ_ϵ, the support of $\mu(\omega_0)$ is a continuous function (with range, the set of unordered h-tuples of points of K). Hence $\Lambda \cap \Gamma_\epsilon \times K$ is an open set and therefore measurable. But $\Lambda - \Lambda \cap \Gamma_\epsilon \times K$ has measure $\leqq \epsilon$ so that Λ is measurable. This type of argument may be used to verify a number of measurability assertions that we shall make. Now Λ must also have measure 0, by applying (9) to the indicator function of Λ. But $(\omega_0 \times K) \cap (\Omega - \Lambda)$ has $\leqq h$ points for every ω_0, and so $\Phi^{-1}(\omega_0) \cap (\Omega - \Lambda)$ has $\leqq h$ points so that (Ω, T, μ) is a finite extension of (Ω_0, T_0, μ). It should be pointed out that the set of points (ω_0, ζ) with ζ in the support of $\mu(\omega_0)$ may be much smaller than the support of μ. Thus if each $\mu(\omega_0)$ is concentrated at one point $\zeta = \phi(\omega_0)$, the support of μ will be the closure of $\{(\omega_0, \phi(\omega_0))\}$ which may be all of Ω.

Now suppose that μ is an ergodic invariant measure for T, that does not define a finite extension of (Ω_0, T_0, μ_0); then a.e., $\mu(\omega_0)$ has infinite support. We claim that in this case, $\mu(\omega_0)$ is, a.e., a non-atomic measure, i.e. it has no discrete component. Otherwise define $\Delta_\delta(\omega_0) = \{(\omega_0, \zeta) : \mu(\omega_0)(\{\zeta\}) > \delta\}$ for $\delta > 0$, and let $\Delta_\delta = \bigcup_{\omega_0 \in \Omega_0} \Delta_\delta(\omega_0)$. Again one can show that Δ_δ is measurable by considering $\Delta_\delta \cap (\Gamma \times K)$ where Γ is compact and $\mu(\omega_0)$ is continuous when restricted to Γ. Now (10) shows that Δ_δ is invariant under T; hence Δ_δ has either measure 0 or 1. This in turn implies that almost all $\Delta_\delta(\omega_0)$ have measure 0 or 1. The latter would imply that $\mu(\omega_0)$ has its support on $\leqq \delta^{-1}$ points of K contrary to our assumption. Hence $\Delta_\delta(\omega_0)$ has measure 0 with probability 1, which implies that $\Delta_\delta(\omega_0)$ is a.e. empty. Letting $\delta \to 0$ shows that $\mu(\omega_0)$ is a.e. non-atomic.

Under the assumption that an ergodic, invariant μ exists that does not define a finite extension, one can show that no other invariant measure ν on (Ω, T) can define a finite extension. For suppose that ν is an ergodic measure with $\nu(\omega_0)$ having its support on $\leqq h$ points for almost all ω_0. The unordered h-tuple $\{\zeta_1(\omega_0), \cdots, \zeta_h(\omega_0)\}$ on which $\nu(\omega_0)$ has its support is a measurable function of ω_0. Note that h is independent of ω_0, since the set of ω_0 having h points in the support of $\mu(\omega_0)$ is invariant under T_0 by (10). Now consider for each ω_0, the set of points

$$\Delta(\omega_0) = \{\zeta \in K : \mu(\omega_0)(\{\zeta_i(\omega_0), \zeta\}) \leqq \epsilon \text{ for some } i = 1, \cdots, h\},$$

and take $\Delta = \bigcup_{\omega_0 \in \Omega_0} (\omega_0 \times \Delta(\omega))$. Again Δ is a measurable subset of Ω, and by (10), Δ is invariant under T. But by (9), $\epsilon \leqq \mu(\Delta) \leqq h\epsilon$ contrary to the assumption that μ is ergodic, if $0 < \epsilon < h^{-1}$. This shows that if an ergodic infinite extension exists, every extension is infinite.

Now suppose that μ and μ' are two invariant measures on (Ω, T), with $\mu(\omega_0)$ and $\mu'(\omega_0)$ non-atomic a. e. Let $\lambda = \frac{1}{2}(\mu + \mu')$ so that $\lambda(\omega_0)$ is again non-atomic a. e. Fix a point $\zeta_0 \in K$ and consider the map Ψ from $\Omega = \Omega_0 \times K \to \Omega' = \Omega_0 \times K'$ (K' is the unit circle in the complex plane) given by

$$\Psi(\omega_0, \zeta) = (\omega_0, e^{2\pi i \lambda(\omega_0)(\{\zeta_0, \zeta\})}). \tag{11}$$

As in the proof of Theorem 1.3, we have that if $\zeta_1, \zeta_2, \zeta_3 \in K$,

$$\lambda(\omega_0)(\{\zeta_1, \zeta_2\}) + \lambda(\omega_0)(\{\zeta_2, \zeta_3\}) \equiv \lambda(\omega_0)(\{\zeta_1, \zeta_3\}) \pmod 1.$$

Hence we have

$$\begin{aligned}
\Psi(T(\omega_0, \zeta)) &= (T_0\omega_0, e^{2\pi i \lambda(T_0\omega_0)(\{\zeta_0, g(\omega_0, \zeta)\})}) \\
&= (T_0\omega_0, e^{2\pi i \lambda(T_0\omega_0)(\{\zeta_0, g(\omega_0, \zeta_0)\})}\, e^{2\pi i \lambda(T_0\omega_0)(\{g(\omega_0, \zeta_0), g(\omega_0, \zeta)\})}) \\
&= (T_0\omega_0, g(\omega_0) e^{2\pi i \lambda(\omega_0)(\{\zeta_0, \zeta\})})
\end{aligned}$$

where $g(\omega_0) = e^{2\pi i \lambda(T_0\omega_0)(\{\zeta_0, g(\omega_0, \zeta_0)\})}$, the last equality being a consequence of (10). Hence, taking $S(\omega_0, \zeta) = (T_0\omega_0, g(\omega_0)\zeta)$, we have

$$\Psi T(\omega_0, \zeta) = S\Psi(\omega_0, \zeta). \tag{12}$$

Here it can be shown that the mappings Ψ and S are measurable. In fact, on a compact set Γ relative to which $\lambda(\omega_0)$ and $\lambda(T_0\omega_0)$ are continuous, the functions $\lambda(\omega_0)(\{\zeta_0, \zeta\})$ and $\lambda(T_0\omega_0)(\{\zeta_0, g(\omega_0, \zeta_0)\})$, entering in the definition of Ψ and S, are actually continuous. This is so because $\lambda(\omega_0)$ is non-atomic.

It is clear from (9) and (11) that Ψ takes the measure λ on Ω into the product measure $\mu_0 \times m$ on Ω', in the sense that $\mu_0 \times m(\Delta) = \lambda(\Psi^{-1}(\Delta))$, or,

equivalently, that $\lambda(f \circ \Psi) = \mu_0 \times m(f)$. We write $\Psi(\lambda) = \mu_0 \times m$. Let μ_* and μ_*' denote respectively $\Psi(\mu)$ and $\Psi(\mu')$, i.e. the images of μ and μ' under the map Ψ. We have then $\mu_0 \times m = \frac{1}{2}(\mu_* + \mu_*')$. But $T\mu = \mu$ implies $\Psi(T\mu) = \Psi(\mu)$ so that by (12), $S(\Psi(\mu)) = \Psi(\mu)$, or $S\mu_* = \mu_*$. Similarly $S\mu_*' = \mu_*'$. Thus S is not an ergodic transformation for $\mu \times m$ unless $\mu_* = \mu_*'$.

Assuming $\mu_* \neq \mu_*'$ it follows that S is not ergodic. The Fourier series argument of Lemma 2.1 applies as well here (for $g(\omega_0)$ measurable rather than continuous), and we deduce that for some $k \neq 0$,

$$g^k(\omega_0) = R(T_0\omega_0)/R(\omega_0).$$

For every interval $\Delta \subset K'$, we may define

$$\Lambda'(\Delta) = \{(\omega_0, \zeta) \in \Omega' : R(\omega_0)^{-1}\zeta^k \in \Delta\}.$$

It is readily verified that $\mu_0 \times m(\Lambda'(\Delta)) = m(\Delta)$. Now $\Lambda'(\Delta)$ is invariant under S, since if $R(\omega_0)^{-1}\zeta^k \in \Delta$ then $R(T_0\omega_0)^{-1}g_0{}^k(\omega_0)\zeta^k \in \Delta$ which is just the condition for $S(\omega_0, \zeta) \in \Lambda'(\Delta)$. Since $m(\Delta)$ is arbitrary, it follows that S has invariant sets of all measures between 0 and 1. From (12) it follows that if $\Lambda'(\Delta)$ is invariant under S, then $\Lambda(\Delta) = \Psi^{-1}(\Lambda'(\Delta))$ is invariant under T. Since Ψ takes λ into $\mu_0 \times m$ it follows that there are invariant sets on Ω of arbitrary λ-measure. But we assumed that μ and μ' were ergodic measures and $\lambda = \frac{1}{2}(\mu + \mu')$. If a measurable set is invariant under T, it must have μ-measure 0 or 1 and μ'-measure 0 or 1. Hence it must have λ-measure 0, $\frac{1}{2}$, or 1. Thus we are led to a contradiction which means that $\mu_* = \mu_*'$.

From this we shall deduce that $\mu = \mu'$. For this it suffices to show that $\mu(\omega_0) = \mu'(\omega_0)$ a.e. But clearly the relative measures of μ and μ' are mapped into the relative measures of μ_* and μ_*' by Ψ. Since $\mu_* = \mu_*'$, we have $\mu_*(\omega_0) = \mu_*'(\omega_0)$ a.e. To complete the proof it suffices to show that the relative measure $\mu(\omega_0)$ is determined a.e. by its image $\mu_*(\omega_0)$ and similarly for μ'. Here we are considering the relative measure $\mu(\omega_0)$ on $\omega_0 \times K$, and by its image we mean the image under Ψ on $\omega_0 \times K'$. Now $\mu(\omega_0)$ and $\mu'(\omega_0)$ are both absolutely continuous with respect to $\lambda(\omega_0)$ which is carried onto lebesgue measure on $\omega_0 \times K'$. If an interval and a subinterval of $\omega_0 \times K$ both map onto the some interval af $\omega_0 \times K'$ under Ψ, the $\lambda(\omega_0)$-measure of the difference must be 0, by (11), and a foriori the $\mu(\omega_0)$- and $\mu'(\omega_0)$-measures will be 0. It follows that the $\mu(\omega_0)$-measure of any interval of $\omega_0 \times K$ is the same as that of the preimage of the image of the interval under Ψ, and hence

the same as the $\mu_*(\omega_0)$-measure of its image under Ψ. Consequently $\mu(\omega_0)$ is determined by $\mu_*(\omega_0)$ and similarly for $\mu'(\omega_0)$. Therefore $\mu=\mu'$.

This completes the proof of the theorem. For we have shown that if there exists some infinite ergodic extension (Ω, T, μ) of (Ω_0, T_0, μ_0) then no ergodic extensions can be finite. In addition we have just shown that any other ergodic extension must be identical with the first. Thus unless T is strictly ergodic, all ergodic extensions must be finite.

The simplest example of this theorem for dimension >1 is for $\Omega=K^2$ and $T\zeta_1=e^{2\pi i\alpha}\zeta_1$, $T\zeta_2=g(\zeta_1,\zeta_2)$. Our theorem asserts that there are two alternatives. Either T is strictly ergodic or all invariant measures on K^2 define finite extensions of (K, S_α, m) where $S_\alpha\zeta=e^{2\pi i\alpha}\zeta$. The latter alternative implies in particular that for some $h>0$ there exists a measurable function $\Delta(\zeta_1)$ with its range in the space of unordered h-tuples of points on K, and with $\Delta(e^{2\pi i\alpha}\zeta)=g(\zeta,\Delta(\zeta))=\{g(\zeta,\zeta'):\zeta'\in\Delta(\zeta)\}$. In particular for $h=1$, this would mean the existence of a solution to $R(e^{2\pi i\alpha}\zeta)=g(\zeta,R(\zeta))$. The set $\Delta(\zeta)$ is here simply the support of the relative measure $\mu(\zeta)$ defined by the extension. Needless to say, in practice it is difficult to ascertain which of the two alternatives actually occurs. Theorem 2.1 gives the only cases for which we have resolved this question.[1]

5. A fixed point theorem and existence of an almost periodic subprocess.

5.1. We restrict our attention once again to the two-dimensional torus K^2. In the last section we saw that if the transformation T on K^2 has a certain form then some assertions could be made regarding the strict ergodicity of T. We now inquire when a general transformation T of K^2 can, by an appropriate choice of coordinates, be put into the specified form. We are not able to fully answer this question; however we can give conditions on T under which it will almost have the required form. To be precise, the condition on T in the last section was that it define an extension of a 1-dimensional process (K, S_α, m), and that one of the two coordinate functions on K^2 can be taken for the coordinate function on K. What we shall obtain are sufficient conditions for the existence of a subprocess (K, S_α, m) for (K^2, T, μ), but the coordinate function for K may not be a possible coordinate function for K^2. The reason is that a function from K^2 to K will be a coordinate function for

[1] *Added in proof*: There is a corollary to the proof of this theorem which is worth noting. In case $T\zeta_1=e^{2\pi i\alpha}\zeta_1$, $T\zeta_2=g(\zeta_1,\zeta_2)$ does not define a finite extension of (K, S_α, m), then eq. (12) establishes a measure theoretic isomorphism between the process in question and the "standard" process of eq. (2) $(r=2)$ in § 2.1.

K^2 only if its level sets are homeomorphic to a circle, and this need not happen for the mapping function from K^2 to K in our case.

If a process (K^2, T, μ) extend some (K, S_α, m) then by Definition 4, the associated map $\phi\colon K^2 \to K$ satisfies $\phi T = S_\alpha \phi$. Looking at ϕ as a function in $C(K^2)$, ϕT is the function we have denoted $T\phi$. Moreover, $S_\alpha \phi = e^{2\pi i \alpha}\phi$, so that (K^2, T, μ) extends (K, S_α, m) only if there exists a solution to $T\phi = e^{2\pi i\alpha}\phi$, i. e. a non-constant eigenfunction of T, in $C(K^2)$. Conversely if there exists such an eigenfunction ϕ, then the algebra of functions of ϕ is invariant under T and so determines a subprocess which will be isomorphic to (K, S_α, m). The main result of this section is the following:

THEOREM 5.1. *If T is a homeomorphism of K^2 leaving no proper closed subset of K^2 invariant and if T is not homotopic to the identity transformation then there exists a continuous eigenfunction ϕ: $T\phi = e^{2\pi i\alpha}\phi$ where α is irrational.*

For the proof of this we shall require two preliminary results, the first of which is a fixed point theorem which is of some independent interest.

LEMMA 5.1. *Let $S\colon K^2 \to K^2$ be a homeomorphism of the form $S\zeta_1 = \zeta_1 e^{2\pi i G(\zeta_1,\zeta_2)}$, $S\zeta_2 = \zeta_1{}^d \zeta_2 e^{2\pi i H(\zeta_1,\zeta_2)}$ where G and H are continuous real valued functions on K^2. Then if $d \neq 0$ and $\sup_{\omega',\omega'' \in K^2} |G(\omega') - G(\omega'')| \geqq 7$, then the transformation S must have a fixed point.*

Remark. The form of S as given here is the most general one for which S is homotopic to the linear map $\bar{S}\zeta_1 = \zeta_1$, $\bar{S}\zeta_2 = \zeta_1{}^d\zeta_2$. We know from § 3 that up to a change of coordinates, this is the most general form which S can have if it is not to have a fixed point by virtue of the Lefshetz fixed point theorem. The present lemma gives conditions under which a fixed point must exist even if the homotopy class of S does not require the existence of a fixed point.

Proof. Let ρ_1 denote the cycle $\{\lambda_2 = 1\}$ and ρ_2, the cycle $\{\zeta_1 = 1\}$. By a cycle we shall always mean a continuous 1-1 image of K. To prove that there is a fixed point we shall show that for every $\epsilon > 0$ there are points $\omega \in K^2$ with $|S\zeta_1(\omega) - \zeta_1(\omega)| < \epsilon$ and $|S\zeta_2(\omega) - \zeta_2(\omega)| < \epsilon$ simultaneously. By compactness there will have to be a fixed point.

Let $\Gamma = \{\omega : |S\zeta_2(\omega) - \zeta_2(\omega)| < \epsilon\} = \{\omega : |\zeta_1{}^d e^{2\pi i H(\zeta_1,\zeta_2)} - 1| < \epsilon\}$. On $K^2 - \Gamma$, the function $\zeta_1{}^d e^{2\pi i H(\zeta_1,\zeta_2)}$ does not take on the value 1. Hence it is an inessential map and the function $\zeta_1{}^d$ which is homotopic to it is also inessential. Since $d \neq 0$ it follows that ζ_1 is also inessential on $K^2 - \Gamma$. Consequently on any cycle in Γ close to the boundary of Γ, ζ_1 is inessential. There

are now two cases to consider. Suppose first that there is some homologically non-trival (relative to K^2) cycle in Γ arbitrarily close to the boundary of Γ. Since ζ_1 is inessential on this cycle, the cycle must be homologous to a multiple of ρ_2. But a cycle homologous to $l\rho_2$ with $|l| > 1$, contains a cycle homologous to ρ_2, so that there is in Γ a cycle homologous to ρ_2. If, on the other hand, all cycles in Γ sufficiently close to the boundary of Γ are homologous to 0 in K^2, it is easily seen that Γ contains a cycle in every homology class and in particular one homologous to ρ_2. Thus, in either case, Γ contains a cycle γ homologous to ρ_2.

To show that there is a point ω in K^2 with $|S\zeta_1(\omega) - \zeta_1(\omega)| < \epsilon$, $|S\zeta_2(\omega) - \zeta_2(\omega)| < \epsilon$, it suffices to show that $|e^{2\pi i G(\omega)} - 1|$ becomes $< \epsilon$ on γ. For this it suffices to show that for arbitrarily small $\delta > 0$, $G(\omega)$ oscillates on γ by an amount $> 1 - \delta$. Now suppose that on γ, $G(\omega)$ oscillates by $\leqq 1 - \delta$ for some $\delta > 0$. Let $[a, b]$ be the range of $G(\omega)$ on K^2, so that $b - a \geqq 7$, and let $[a', b'] \subset [a, b]$ be the range of $G(\omega)$ on γ, so that $b' - a' \leqq 1 - \delta$. One of the intervals (a, a') or (b', b) has length $\geqq 3 + \delta/2$. Assume it is the first, the argument being entirely analogous if it is the second, so that $a' - a \geqq 3 + \delta/2$. Let Δ be the set of points on K^2 for which $G(\omega)$ lies in (a, a'). We shall show that Δ contains a cycle homologous to ρ_1. Since any cycle homologous to ρ_1 intersects any cycle homologous to ρ_2 it will follow that γ must intersect Δ contrary to the supposition that on γ, $G(\omega)$ omits the values in (a, a'). This contradiction implies that $G(\omega)$ does oscillate by $> 1 - \delta$ on γ and the desired point ω exists.

We now lift the map S to a map S^* of the plane. If we assume that the point $(0, 0)$ in R^2 corresponds to $(1, 1)$ in K^2, then S will have the form

$$\begin{aligned} \theta_1^* &= S^*\theta_1 = \theta_1 + G(\theta_1, \theta_2) \\ \theta_2^* &= S^*\theta_2 = d\theta_1 + \theta_2 + H(\theta_1, \theta_2). \end{aligned} \tag{13}$$

Let Σ denote the strip in R^2 given by $0 \leqq \theta_1 \leqq 1$. Δ lifts to a set Δ^* in the plane on which $G(\theta_1, \theta_2)$ takes values in (a, a_1). Hence $\Delta^* \cap \Sigma$ contains at least those points for which $a + 1 \leqq \theta_1 + G(\theta_1, \theta_2) < a_1$, i.e., the points for which $a + 1 \leqq \theta_1^* < a_1$. To show that Δ contains a cycle homologous to ρ_1, we will prove that for some θ_2 there is a curve joining $(0, \theta_2)$ to $(1, \theta_2)$ in Σ for which $a + 1 < \theta_1^* < a_1$. Such a curve maps onto a cycle homologous to ρ_1 and lying in Δ.

The function θ_1^* is well defined on R^2 and hence if a point (θ_1, θ_2) belongs to the curve $\{\theta_1^* = C\}$, then the point $(\theta_1 + 1, \theta_2)$ cannot also belong to this curve since $\theta_1^*(\theta_1 + 1, \theta_2) = C + 1$ by (13). A less evident fact is that $(\theta_1 + t, \theta_2)$ cannot belong to $\{\theta_1^* = C\}$ if (θ_1, θ_2) does, for any $t \geqq 1$. To

see this we note that if (θ_1,θ_2) and (θ_1+t,θ_2) belong to $\{\theta_1=C\}$ for $t\geqq 1$, then since $\{\theta_1{}^*=C\}$ is connected (since S^* is a homeomorphism) it follows that for some $(\theta_1',\theta_2')\in\{\theta_1{}^*=C\}$, we will also have $(\theta_1'+1,\theta_2')\in\{\theta_1{}^*=C\}$. From this we conclude that on a line segment: $\theta_2=\theta_2'$, $\theta_1'\leqq\theta_1\leqq\theta_1'+1$, the function $\theta_1{}^*$ cannot takes values $\geqq\theta_1{}^*(\theta_1',\theta_2')+2$ or $\leqq\theta_1{}^*(\theta_1,\theta_2')-1$. For if at a point (θ_1,θ_2') in this segment $\theta_1{}^*(\theta_1,\theta_2')\geqq\theta_1{}^*(\theta_1',\theta_2')+2$ then $\theta_1{}^*(\theta_1-1,\theta_2')\geqq\theta_1{}^*(\theta_1'+1,\theta_2')$. But every value of $\theta_1{}^*$ exceeding $\theta_1{}^*(\theta_1'+1,\theta_2')$ occurs at some $(\theta_1'',\theta_2'),\theta_1''>\theta_1'+1$ and $\theta_1''-(\theta_1-1)\geqq 1$ since $\theta_1\leqq\theta_1'+1\leqq\theta_1''$. A similar argument gives the other inequality.

Now in Σ, $G(\theta_1,\theta_2)$ attains its maximum and its minimum, so that $\theta_1{}^*$ must attain a maximum $\geqq b$ and a minimum $\leqq a+1$ in Σ. Also if S^* is a homeomorphism, $\theta_1{}^*$ cannot attain its maximum in Σ at an interior point of Σ. This implies that on the line $\theta_1=0$, $\theta_1{}^*$ attains a minimum $\leqq a+1$ and a maximum $\geqq b-1$. In particular, $\theta_1{}^*$ will vary between $a+1$ and $a+2+\delta/2<b-1$ on $\theta_1=0$. There will therefore be an interval $\theta_2{}^{(1)}\leqq\theta_2\leqq\theta_2{}^{(2)}$ on $\theta_1=0$ on which $\theta_1{}^*$ varies from $a+1+\delta/4$ to $a+2+\delta/4$ as θ_2 goes from $\theta_2{}^{(1)}$ to $\theta_2{}^{(2)}$ and such that neither extreme value of $\theta_1{}^*$ occurs in the interior of this interval.

Let Λ^* be the relatively open set in Σ defined by the condition that $a+1<\theta_1{}^*<a+3+\delta/2$. $\Lambda^*\subset\Delta^*\cap\Sigma$ since $a+3+\delta/2\leqq a_1$. The interval on $\theta_1=0$ from $(0,\theta_2{}^{(1)})$ to $(0,\theta_2{}^{(2)})$ lies in Λ^*; let $\Lambda_0{}^*$ be the connected component of Λ^* containing this interval. We claim that the point $(1,\theta_2{}^{(2)})$, which belongs to Λ^* because

$$\theta_1{}^*(1,\theta_2{}^{(2)})=1+\theta_1{}^*(0,\theta^{(2)})=a+3+\delta/4<a+3+\delta/2,$$

actually belongs to $\Lambda_0{}^*$. Suppose that this were not the case. Then one of the bounding lines $\theta_1{}^*=a+1$ or $\theta_1{}^*=a+3+\delta/2$ must separate $(1,\theta_2{}^{(2)})$ from $(0,\theta_2{}^{(2)})$ and $(0,\theta_2{}^{(1)})$. But on the segment from $(0,\theta_2{}^{(2)})$ to $(1,\theta_2{}^{(2)})$, $\theta_1{}^*$ cannot take a value less than $a+1\ +\delta/4$, since $\theta_1{}^*(0,\theta_2{}^{(2)})=a+2+\delta/4$. Hence the line $\theta_1{}^*=a+1$ cannot separate the two points. On the other hand, the same reasoning shows that $\theta_1{}^*$ cannot exceed $a+3+\delta/4$ on the line segment joining $(0,\theta_2{}^{(1)})$ to $(1,\theta_2{}^{(1)})$. Hence the line $\theta_1{}^*=a+3+\delta/2$ separating $(1,\theta_2{}^{(2)})$ from $\Lambda_0{}^*$ cannot pass below the level $\theta_2=\theta_2{}^{(1)}$. In order to separate $(0,\theta_2{}^{(2)})$ from $(1,\vartheta_2{}^{(2)})$ it must however pass below the level $\theta_2=\theta_2{}^{(2)}$. Hence on one of the lines $\theta_1=0$, 1, it lies between the levels $\theta_2=\theta_2{}^{(1)}$ and $\theta_2{}^{(2)}$. This means that in the interval $\theta_2{}^{(1)}<\theta_2<\theta_2{}^{(2)}$ on $\theta_1=0$, $\theta_1{}^*$ takes on either the value $a+3+\delta/2$ or $a+2+\delta/2$. Both of these are impossible since in the interior of the interval in question $\theta_1{}^*<a+2+\delta/4$.

This shows that $(1,\theta_2^{(2)}) \in \Lambda_0^*$, so that there exists a curve β joining $(0,\theta_2^{(2)})$ to $(1,\theta_2^{(2)})$ lying wholly in $\Lambda^* \subset \Delta^*$. It follows that β maps onto a cycle in K^2 homologous to ρ_1 and lying in Δ. This completes the proof of the lemma.

LEMMA 5.2. *Let T be a transformation of a compact metric space Ω leaving no proper closed subset of Ω invariant. If $f \in C(\Omega)$, then a necessary and sufficient condition for $f = Tg - g$ with $g \in C(\Omega)$, is that the functions $f + Tf + \cdots + T^n f$ are uniformly bounded.*

This lemma appears in [3], p. 135 and, in slightly disguised form, in [2], p. 164.

5.2. *Proof of Theorem* 5.1. From the fact that T has no fixed point we may infer that T has the form of S in Lemma 5.1, for some choice of coordinates (ζ_1, ζ_2). In these coordinates

$$T^n\zeta_1 = \zeta_1 e^{2\pi i(G(T^{n-1}\omega)+\cdots+G(\omega))}$$
$$T^n\zeta_2 = \zeta_1^{nd}\zeta_2 e^{2\pi i H_n(\omega)}$$

for some continuous real valued function H_n, as may be seen by induction. Thus T^n also has the form of S, just with different values of d. Since no power of T has a fixed point if T leaves no proper closed subset of K^2 invariant, Lemma 5.1 implies that for every n, the function $(G(T^{n-1}\omega) + \cdots + G(\omega))$ oscillates by < 7 on K^2. Now for some ω, $\lim n^{-1}(G(T^{n-1}\omega) + \cdots + G(\omega)) = \alpha$ exists by the ergodic theorem applied to some stationary measure on K^2. It follows that this limit exists and equals α for all ω in K^2, and therefore for any invariant probability measure μ, $\mu(G) = \alpha$. Now take $G'(\omega) = G(\omega) - \alpha$; then $\mu(G'(T^{n-1}\omega) + \cdots + G'(\omega)) = 0$. It follows that in K^2, $G'(T^{n-1}\omega) + \cdots + G'(\omega)$ cannot be everywhere strictly positive or strictly negative. Since this expression oscillates by < 7 it follows that

$$|\, G'(T^{n-1}\omega) + \cdots + G'(\omega) \,| < 7$$

for all n. We may now apply Lemma 5.2 since T leaves no proper closed subset of K^2 invariant. We conclude that there exists a continuous function $Q(\omega)$ on K^2 with $G'(\omega) = Q(T\omega) - Q(\omega)$, whence $G(\omega) = Q(T\omega) - Q(\omega) + \alpha$. Hence $T\zeta_1 = \zeta_1 e^{2\pi i[Q(T\omega)-Q(\omega)+\alpha]}$ and $T(\zeta_1 e^{-2\pi i Q(\omega)}) = e^{2\pi i\alpha}(\zeta_1 e^{-2\pi i Q(\omega)})$ so that $\phi(\omega) = \zeta_1 e^{-2\pi i Q(\omega)}$ is the eigenfunction sought. Here α must be irrational since if $\alpha = l/k$, then the set of ω for which $\phi(\omega) = e^{2\pi i\nu/k}$, $\nu = 0, 1, \cdots, k-1$, would be a proper closed subset of K^2. (Since $\phi(\omega)$ is homotopic to ζ_1, it is an essential map and therefore $\phi(\omega)$ takes on all values on the unit circle.) This concludes the proof of the theorem.

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ON THE SUMMABILITY, BY SPHERICAL RIESZ MEANS, OF DERIVED MULTIPLE FOURIER SERIES.*

By A. J. White.

1. Introduction. We suppose, throughout this paper that $f(x) \equiv f(x_1, \cdots, x_k)$ is integrable in the Lebesgue sense in $0 \leqq x_i \leqq 2\pi$ $(i = 1, \cdots, k)$ and is periodic, with period 2π in each variable. The Fourier series of $f(x)$ may be written

$$f(x) \sim \sum_{n_1 \cdots n_k = -\infty}^{\infty} \cdots \sum a_{n_1 \cdots n_k} e^{i(n_1 x_1 + \cdots + n_k x_k)},$$

where

$$a_{n_1 \cdots n_k} = (2\pi)^{-k} \int_{x_1 \cdots x_k = 0}^{2\pi} \cdots \int f(x) e^{-i(n_1 x_1 + \cdots + n_k x_k)} dx_1 \cdots dx_k.$$

We write [1]

$$A_n(x) = \sum_{n = n_1^2 + \cdots + n_k^2} a_{n_1 \cdots n_k} e^{i(n_1 x_1 + \cdots + n_k x_k)},$$

with the convention that $A_n(x) \equiv 0$ if n cannot be expressed as the sum of k squares of integers. We form $B_n(x)$ from the series

$$\sum_{n_1 \cdots n_k = -\infty}^{\infty} \cdots \sum b_{n_1 \cdots n_k} e^{i(n_1 x_1 + \cdots + n_k x_k)},$$

where the $b_{n_1 \cdots n_k}$ are arbitrary complex numbers, in a similar way.

The spherical mean, $f_{x^0}(t)$ of the function $f(x)$ at the point $x^0 \equiv (x_1^0, \cdots, x_k^0)$ is defined by

$$f_{x^0}(t) = \frac{\Gamma(\frac{1}{2}k)}{2(\pi)^{\frac{1}{2}k}} \int_\sigma f(x_1^0 + t\xi_1, \cdots, x_k^0 + t\xi_k) \, d\sigma_\xi \qquad (k \geqq 2)$$

where σ is the sphere $\xi_1^2 + \cdots + \xi_k^2 = 1$ and $d\sigma_\xi$ its $(k-1)$-dimensional volume element. If $k = 1$ then

$$f_{x^0}(t) = \tfrac{1}{2}\{f(x^0 + t) + f(x^0 - t)\}.$$

Riesz means $S_R^\delta(f, x)$, of order δ and type n, of the series $\sum_{n=0}^{\infty} A_n(x)$ are defined by

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[1] The theory of spherical summability of multiple Fourier series was initiated by Bochner [2]. An introduction and an account of the main results is given in [7].

$$S_R^0(f,x) = S_R(f,x) = \sum_{n \leqq R^2} A_n(x)$$

$$S_R^\delta(f,x) = 2\delta/R^2 \int_0^R (1-u^2/R^2)^{\delta-1} u S_u(f,u)\,du \qquad (\delta > 0)$$

$$= \sum_{n \leqq R^2} (1-n/R^2)^\delta A_n(x). \qquad (\delta > 0)$$

The Laplacian Δ_x^q, defined by

$$\Delta_x^1 = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_k^2}$$

$$\Delta_x^{r+1} = \Delta_x^1(\Delta_x^r) \qquad (r = 1, 2, \cdots)$$

applied to the terms of the series $\sum_{n=0}^{\infty} A_n(x)$ yields

$$(-)^q \sum_{n=1}^{\infty} n^q A_n(x). \tag{1.1}$$

Observing that

$$\Delta_x^q S_R^\delta(f,x) = (-)^q \sum_{n \leqq R^2} (1-n/R^2)^\delta n^q A_n(x),$$

we may state known results concerning the summability, by Riesz means, of the series (1.1), in the following form.

THEOREM A.[2] *If there exists a polynomial $P(t)$, in t^2, of degree $2(q-1)$ $(\geqq 0)$ such that*

$$f_x(t) - P(t) \sim \frac{s t^{2q} \Gamma(\tfrac{1}{2}k)}{2^{2q} q!\, \Gamma(q + \tfrac{1}{2}k)},$$

as $t \to +0$, then

$$\Delta_x^q S_R^\delta(f,x) \to s$$

as $R \to \infty$, whenever $\delta > 2q + \tfrac{1}{2}(k-1)$.

THEOREM B.[3] *Suppose that the trigonometric series $\sum_{n=0}^{\infty} B_n(x)$ is summable*[4] *(n,δ) to s, for a non-negative integer δ, and that, for some $\epsilon > 0$, $B_n(0) = o(n^{\delta-\epsilon})$. Then if q $(> \delta + 1)$ is a positive integer, and if*

$$g(x) = \frac{B_0(x_1 + \cdots + x_k)^{2q}}{k^q (2q)!} + (-)^q \sum_{n=1}^{\infty} n^{-q} B_n(x),$$

[2] Theorems A and B are given by Shapiro ([9] Theorems 2, 3 and 5). We have altered his notation slightly.

[3] Shapiro has obtained a sharper version of this Theorem (see [10] Theorem 2).

[4] I. e. $\sum_{n \leqq R^2} (1-n/R^2)^\delta B_n(x) \to s$ as $R \to \infty$.

there exists a polynomial $P(t)$, *in* t^2, *of degree* $2(q-1)$ *such that*

$$g_x(t) - P(t) \sim \frac{st^{2q}\Gamma(\frac{1}{2}k)}{2^{2q}q!\Gamma(\frac{1}{2}k+q)},$$

as $t \to +0$.

In section 3 of this paper we obtain, in Theorem 2, a generalization of Theorem A. We also obtain (Theorem 1) a converse result, which is a version of Theorem B applying specifically to multiple Fourier series.

We also extend results recently obtained in [13] and show that the summability, by spherical Riesz means, of a derived multiple Fourier series depends on the behaviour of a certain one-dimensional Fourier series. Known results then enable us (at least when k is odd) to give a complete solution to the spherical summability problem for derived multiple Fourier series.

2. Preliminaries. In this section we summarise known results, obtain some lemmas, and give definitions and notation which, together with that already given in section 1, will be required later.

Bessel Functions. [11]. The Bessel function $J_\mu(t)$, of the first kind and order μ $(\geqq 0)$, is defined by

$$J_\mu(t) = (t/2)^\mu \sum_{\nu=0}^{\infty} (-)^\nu \frac{t^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)}.$$

We write, throughout this paper,

$$V_\mu(t) = t^{-\mu}J_\mu(t) \qquad (t > 0)$$

$$V_\mu(0) = 1.$$

It is known that

$$(2.1) \qquad |V_\mu(t)| \leqq \begin{cases} A(\mu) & 0 \leqq t < 1 \\ A(\mu)t^{-\mu-\frac{1}{2}} & t \geqq 1, \end{cases}$$

and that

$$(2.2) \qquad V_\mu'(t) = -tV_{\mu+1}(t).$$

Fractional Integrals. For $t > 0$, $\zeta \geqq 1$, $u^{\zeta-1}\phi(u) \in L(0,t)$ we write

$$\left.\begin{aligned} \Phi_{p,\zeta}(t) &= \frac{1}{2^{p-1}\Gamma(p)}\int_0^t (t^2-u^2)^{p-1}u^{\zeta-1}\phi(u)\,du \\ \phi_{p,\zeta}(t) &= \frac{2^p\Gamma(p)}{B(p,\frac{1}{2}\zeta)}t^{-2p-\zeta+2}\Phi_{p,\zeta}(t) \end{aligned}\right\}(p > 0)$$

$$\phi_{0,\zeta}(t) = \Phi_{0,\zeta}(t) = \phi(t),$$

and a similar notation is used when we replace $\phi(u)$ by $g(u)$ etc., with the convention that if $\phi(u)=f_x(u)$ and $\zeta=k$ we write $f_{x,p}(u)$ for $\phi_{p,\zeta}(u)$.

If, for $q\geqq 0$,

$$\Phi_{p,\zeta}(t)\sim\frac{B(p,q+\frac{1}{2}\zeta)}{2^p\Gamma(p)}st^{2p+2q+\zeta-2}$$

as $t\to+0$ we write [5]

$$\phi(t)\sim st^{2q}(C_\zeta,p)$$

as $t\to+0$.

It is known that [6] if $p\geqq 1$

$$|f_x(t)|=O(1)\,(C_k,p) \tag{2.3}$$

as $t\to\infty$.

Summability by Riesz means. If $\delta\geqq 0$, and if

$$\sum_{n\leqq R^2}(1-n/R^2)^\delta\alpha_n\to s$$

as $R\to\infty$, then $\sum\alpha_n$ is said to be summable (n,δ) to s.

Bochner's Integral. Bochner [7] has shown that if $\delta>\frac{1}{2}(k-1)$ then

$$S_R{}^\delta(f,x)=\frac{2^{\delta-\frac{1}{2}k+1}\Gamma(\delta+1)}{\Gamma(\frac{1}{2}k)}R^k\int_0^\infty f_x(t)t^{k-1}V_{\delta+\frac{1}{2}k}(tR)\,dt.$$

Since, by (2.3), $|f_x(t)|=O(1)(C_k,1)$ as $t\to\infty$ it follows, using (2.1) that

$$\begin{aligned}\left|\int_\pi^\infty f_x(t)t^{k-1}V_{\delta+\frac{1}{2}k}(tR)\,dt\right| &\leqq AR^{-\delta-\frac{1}{2}k-\frac{1}{2}}\int_\pi^\infty|f_x(t)|t^{k-1}\frac{dt}{t^{\delta+\frac{1}{2}k+\frac{1}{2}}}\\ &=AR^{-\delta-\frac{1}{2}k-\frac{1}{2}}\left\{\left[\frac{O(t^k)}{t^{\delta+\frac{1}{2}k+\frac{1}{2}}}\right]_\pi^\infty+\int_\pi^\infty\frac{O(t^k)}{t^{\delta+\frac{1}{2}k+\frac{3}{2}}}\,dt\right\}\\ &=O(R^{-k-\alpha}),\end{aligned}$$

where $\alpha\geqq 0$, provided that $\delta>\alpha+\frac{1}{2}k-\frac{1}{2}$. Hence, in the case $k=1$,

$$S_R{}^\delta(f,x)=\frac{2^{\delta+\frac{1}{2}}\Gamma(\delta+1)}{\Gamma(\frac{1}{2})}R\int_0^\pi f_x(t)V_{\delta+\frac{1}{2}}(tR)\,dt+o(R^{-\beta}), \tag{2.4}$$

where $\beta\geqq 0$, provided that $\delta>\beta$.

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[5] If $\phi(u)\in L$ this is equivalent to $\phi(t)\sim st^{2q}(C,p)$ see [7].

[6] Bochner [2], page 185, (2.3) means

$$\int_0^t(t^2-u^2)^{p-1}u^{k-1}\,|f_x(u)|\,du=O(t^{2p+k-2})\quad(t\to\infty).$$

[7] Bochner [2] page 176, Formula 10.

We require also the following lemmas.

LEMMA 1. *If r is zero or a positive integer, and if $\sum_{n=1}^{\infty} n^r a_n$ is summable $(n, \delta + r)$ $(\delta \geqq 0)$ then $\sum_{n=1}^{\infty} a_n$ is summable (n, δ).*

This follows from the corresponding result, given by Bosanquet,[8] for summability (C), and the equivalence of the Riesz and Cesaro processes.

LEMMA 2.[9] *If h is zero or a positive integer, and if*

$$\lambda(u) = \sum_{\nu=q}^{\infty} (-)^{\nu} \frac{u^{2\nu-2q}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)} \quad (\mu \geqq 0)$$

then

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^h \lambda(xt)\right| \leqq \begin{cases} At^{2h} & (0 < x < t^{-1}) \\ At^{2h}[(xt)^{-\mu-2q-h-\frac{1}{2}} + (xt)^{-2h-2}] & (x \geqq t^{-1}), \end{cases}$$

where A is independent of x and t.

Firstly for finite z and t

$$\left(\frac{1}{z}\frac{d}{dz}\right)^h \lambda(z) = \sum_{\nu=q+h}^{\infty} (-)^{\nu} \frac{(2\nu-2q)\cdots(2\nu-2q-2h+2)}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)} z^{2\nu-2q-2h},$$

where the differentiation under the summation sign is justified by the uniform convergence of the resulting series. Hence, for $h = 0, 1, \cdots$, $\left(\frac{1}{z}\frac{d}{dz}\right)^h \lambda(z)$ is regular in the whole plane so that there exists a constant K, independent of z, such that

$$\left|\left(\frac{1}{z}\frac{d}{dz}\right)^h \lambda(z)\right| < K \qquad (|z| < 1). \tag{2.5}$$

Next, for $u > 0$,

$$\lambda(u) = 2^{\mu} u^{-2q} V_{\mu}(u) - P(u)$$

where

$$P(u) = \sum_{\nu=0}^{q-1} (-)^{\nu} \frac{u^{2\nu-2q}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)}.$$

Now by (2.1) and (2.2), for $u \geqq 1$, $h = 0, 1, \cdots$,

$$\left|\left(\frac{1}{u}\frac{d}{du}\right)^h u^{-2q} V_{\mu}(u)\right| \leqq A u^{-\mu-2q-h-\frac{1}{2}}, \tag{2.6}$$

where A is independent of u. Further, for $u \geqq 1$,

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[8] Bosanquet [4] page 277, Lemma 3.

[9] Cf. also Shapiro [9], [10].

$$|(\frac{1}{u}\frac{d}{du})^h P(u)| = \begin{cases} |P(u)| & (h=0) \\ |\sum_{\nu=0}^{q-1}(-)^\nu \frac{(2\nu-2q)\cdots(2\nu-2q-2h+2)}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)} u^{2\nu-2q-2h}| & (h=1,2,\cdots) \end{cases} \tag{2.7}$$

$$\leqq Au^{-2h-2} \qquad (h=0,1,\cdots).$$

Combining (2.6) and (2.7) we obtain, for $u \geqq 1$ and $h = 0, 1, \cdots$,

$$|(\frac{1}{u}\frac{d}{du})^h \lambda(u)| \leqq Au^{-2q-h-\mu-\frac{1}{2}} + Au^{-2h-2}. \tag{2.8}$$

The required inequalities follow from (2.5) and (2.8).

Lemma 3. *If $p > \frac{1}{2}(k+1)$ then for $t > 0$ and all x,*

$$f_{x,p}(t) = 2^{p+\frac{1}{2}(k-2)}\Gamma(p+\tfrac{1}{2}k)\sum_{n=0}^{\infty} A_n(x)\, V_{p+\frac{1}{2}(k-2)}(t\sqrt{n}).$$

There exists a sequence $\{f^r(x)\}$ $(r = 1, 2, \cdots)$ of exponential polynomials such that

$$\lim_{r\to\infty} \int_{x_1\cdots x_k=0}^{2\pi}\cdots\int |f(x) - f^r(x)|\, dx_1\cdots dx_k = 0. \tag{2.9}$$

If

$$f^r(x) = \sum_{n=0}^{N_1(r)} b_{n_1\cdots n_k}{}^{(r)} e^{i(n_1x_1+\cdots+n_kx_k)} \tag{2.10}$$

then it follows from (2.9) that

$$\lim_{r\to\infty} b_{n_1\cdots n_k}{}^{(r)} = a_{n_1\cdots n_k} \tag{2.11}$$

uniformly in $(n_1, \cdots, n_k)$ and that, for each fixed positive t and all x

$$\lim_{r\to\infty} f_{x,p}{}^r(t) = f_{x,p}(t). \tag{2.12}$$

If $N_1(r)$ does not tend to infinity with r it follows that $f(x)$ differs from an exponential polynomial only in a set of (k-dimensional) measure zero. In fact in this case the Fourier series of $f(x)$ is finite and we may write

$$f(x) = \sum_{n=0}^{N_0} A_n(x) = \sum_{n=0}^{N_0} a_{n_1\cdots n_k} e^{i(n_1x_1+\cdots+n_kx_k)} \tag{2.13}$$

p. p. in $0 \leqq x_i \leqq 2\pi$ $(i = 1, \cdots, k)$. The result stated then follows on taking

the spherical mean of order p of both sides of (2.13), since[10] the spherical mean of order p of $e^{i(n_1x_1+\cdots+n_kx_k)}$ is, at the point x,

$$2^{p+\frac{1}{2}(k-2)}\Gamma(p+\tfrac{1}{2}k)\,V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})\,e^{i(n_1x_1+\cdots+n_kx_k)}.$$

Now suppose that $N_1(r)\to\infty$ as $r\to\infty$. Taking the spherical mean of order p of both sides of (2.9) we have

$$f_{x,p}{}^r(t)=c\sum_{n=0}^{N_1(r)} b_{n_1\cdots n_k}{}^{(r)}e^{i(n_1x_1+\cdots+n_kx_k)}V_{p+\frac{1}{2}(k-2)}(t\sqrt{n}),$$

where $c=2^{p+\frac{1}{2}(k-2)}\Gamma(p+\frac{1}{2}k)$. Consequently, writing

$$T_N=c\sum_{n=0}^{N} a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}V_{p+\frac{1}{2}(k-2)}(t\sqrt{n}),$$

we have, for fixed r and $N>N_1(r)$

$$\begin{aligned}T_N-f_{x,p}(t)&=c\sum_{n=0}^{N_1(r)}(a_{n_1\cdots n_k}-b_{n_1\cdots n_k}{}^{(r)})\,e^{i(n_1x_1+\cdots+n_kx_k)}V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})\\ &\quad+c\sum_{n=N_1(r)+1}^{N} a_{n_1\cdots n_k}e^{i(n_1x_1+\cdots+n_kx_k)}V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})\\ &\quad+f_{x,p}{}^r(t)-f_{x,p}(t).\end{aligned}\tag{2.14}$$

Since $a_{n_1\cdots n_k}=o(1)$ as $(n_1,\cdots,n_k)\to(\infty,\cdots,\infty)$ and since the number of points $(n_1,\cdots,n_k)$ for which $n_1{}^2+\cdots+n_k{}^2=n$ is $O(n^{\frac{1}{2}k+\zeta-1})$ where $\zeta(>0)$ is arbitrary, it follows from (2.1) that $\sum_{n=0}^{\infty}a_{n_1\cdots n_k}V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})$ is absolutely convergent for fixed $t(>0)$ provided that $p>\frac{1}{2}(k+1)+2\zeta$. Hence, from (2.1), (2.11), (2.12) and (2.14), for fixed $t(>0)$ and all x,

$$\begin{aligned}\overline{\lim_{N\to\infty}}\,|T_N-f_{x,p}(t)|&\leqq c\sum_{n=0}^{N_1(r)}|a_{n_1\cdots n_k}-b_{n_1\cdots n_k}{}^{(r)}|\,|V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})\\ &\quad+c\sum_{N_1(r)+1}^{\infty}|a_{n_1\cdots n_k}|\,|V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})|\\ &\quad+|f_{x,p}{}^r(t)-f_{x,p}(t)|\\ &=o\{\sum_{n=1}^{\infty}n^{\frac{1}{2}k+\zeta-1}|V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})|\}\\ &\quad+O\{\sum_{N_1(r)+1}^{\infty}n^{\frac{1}{2}k+\zeta-1}|V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})|\}\\ &\quad+o(1)\\ &=o(1)\end{aligned}$$

as $r\to\infty$, provided that $p>\frac{1}{2}(k+1)+2\zeta$.

[10] See, for example, [6] page 994, Formula 10.

Hence

$$\lim_{N\to\infty} T_N = f_{x,p}(t),$$

for fixed positive t and all x provided that $p > \frac{1}{2}(k+1) + 2\zeta$. Since $\zeta(>0)$ is arbitrary the result stated follows immediately.

It is perhaps worth remarking that Lemma 3 may be put into a different form. We have, for $p > \frac{1}{2}(k+1)$ and $t > 0$,

$$\begin{aligned} f_{x,p}(t) &= c \lim_{N\to\infty} \sum_{n=0}^{N} A_n(x) V_{p+\frac{1}{2}(k-2)}(t\sqrt{n}) \\ &= c \lim_{N\to\infty} \{ \sum_{n=0}^{N} S_{\sqrt{n}}(f,x) \Delta[V_{p+\frac{1}{2}(k-2)}(t\sqrt{n})] \\ &\qquad + S_{\sqrt{N}}(f,x) V_{p+\frac{1}{2}(k-2)}(t\sqrt{N}) \} \end{aligned}$$

where $S_R(f,x) = S_R{}^0(f,x) = \sum_{n\leqq R^2} A_n(x)$.

Hence, since $S_R(f,x) = O(R^{k+\zeta})$ (where $\zeta(>0)$ is arbitrary) it follows, using (2.1) that for fixed $t(>0)$

$$\begin{aligned} f_{x,p}(t) &= c \lim_{N\to\infty} \{ t^2 \int_0^{\sqrt{N}} R S_R(f,x) V_{p+\frac{1}{2}(k-2)+1}(tR)\, dR \\ &\qquad + \frac{o(N^{\frac{1}{2}k+\frac{1}{2}\zeta})}{N^{\frac{1}{2}(p+\frac{1}{2}k-\frac{1}{2})}} \} \\ &= ct^2 \int_0^{\infty} R S_R(f,x) V_{p+\frac{1}{2}k}(tR)\, dR, \end{aligned}$$

provided that $p > \frac{1}{2}(k+1)$. In this form Lemma 3 has been obtained by Chandrasekharan [11] with the condition $p > \frac{1}{2}(k+1)$ replaced by $p > \frac{1}{2}(k+3)$.

3. We suppose, throughout the rest of this paper, that q is a positive integer and that

$$g(t) = \frac{2^{2q} q! \Gamma(q+\frac{1}{2}k)}{\Gamma(\frac{1}{2}k)} \{ f_{x^0}(t) - \Gamma(\tfrac{1}{2}k) \sum_{\nu=0}^{q-1} \frac{a_\nu t^{2\nu}}{2^{2\nu} \nu! \Gamma(\nu+\frac{1}{2}k)} \} \quad (t>0),$$

where the a_ν $(\nu = 0, \cdots, q-1)$ are independent of t, and x^0 is some given point.

We first obtain

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[11] [6] page 996, Lemma 2, the case $\delta = 0$. Another form of this lemma is given in [5] p. 216 (and is reproduced in [7] p. 135) but the proof of this version appears to be invalid.

THEOREM 1. *If* $(-)^q \sum_{n=1}^{\infty} n^q A_n(x^0)$ *is summable* $(n, \delta + 2q)$ $(\delta \geqq 0)$ *to* s *then there exist constants* a_ν *such that*

$$g(t) \sim st^{2q} (C_k, p), \tag{3.1}$$

as $t \to +0$, *whenever* $p > \max[\delta - \tfrac{1}{2}(k-3), \tfrac{1}{2}(k+1)]$.

Moreover, for $\nu = 0, 1, \cdots, q-1$,

$$a_\nu = (-)^\nu \lim_{R \to \infty} \sum_{n \leqq R^2} (1 - n/R^2)^{\delta+q+\nu} n^\nu A_n(x^0). \tag{3.2}$$

We may suppose, without loss of generality, that $s = 0$. For if, in particular, $f(x) = (-)^q s e^{i(x_1 + \cdots + x_k)}$ and $x^0 = (0, \cdots, 0)$ then [12] the derived series (1.1) consists of the single term $s e^{i(x_1 + \cdots + x_k)}$ which converges to s at $x = x^0$. Moreover, if $a_\nu = (-)^{\nu+q} s$ then the a_ν satisfy (3.2). Since the spherical mean at $x = x^0$ of $(-)^q s e^{i(x_1 + \cdots + x_k)}$ is $(-)^q s 2^{\frac{1}{2}(k-2)} \Gamma(\tfrac{1}{2}k) V_{\frac{1}{2}(k-2)}(t)$ it follows that, in this case,

$$\begin{aligned} g(t) &= 2^{2q} q! \Gamma(q + \tfrac{1}{2}k) \{ (-)^q s 2^{\frac{1}{2}(k-2)} V_{\frac{1}{2}(k-2)}(t) - \sum_{\nu=0}^{q-1} \frac{(-)^{\nu+q} t^{2\nu} s}{2^{2\nu} \nu! \Gamma(\nu + \frac{1}{2}k)} \} \\ &= st^{2q} + (-)^q s 2^{2q} q! \Gamma(q + \tfrac{1}{2}k) \sum_{\nu=q+1}^{\infty} (-)^\nu \frac{t^{2\nu}}{2^{2\nu} \nu! \Gamma(\nu + \frac{1}{2}k)} \\ &\sim st^{2q}, \end{aligned}$$

as $t \to +0$. Hence, if $f(x)$ is replaced by $f(x) - (-)^q s e^{i(x_1 - x_1^0 + \cdots + x_k - x_k^0)}$, s is replaced by zero.

Writing

$$T(0, u) = \sum_{1 \leqq n \leqq u^2} n^q A_n(x^0)$$

$$T(m, u) = \int_1^u y T(m-1, y) \, dy \qquad (m = 1, 2, \cdots)$$

we note that since $A_n(x^0) = O(n^{\frac{1}{2}k - 1 + \zeta})$ where $\zeta (> 0)$ is arbitrary

$$T(m, u) = O(u^{2m + 2q + 2\zeta + k}) \qquad (m = 0, 1, \cdots). \tag{3.3}$$

Writing also, for simplicity, $\tau = \delta + 2q$, $\mu = p + \tfrac{1}{2}(k-2)$,

$$\left. \begin{aligned} \gamma(t) &= 2^\mu t^{-2q} V_\mu(t) \\ \Gamma(\mu + 1) \phi(t) &= f_{x^0, p}(t) - A_0 \end{aligned} \right\} \tag{3.4}$$

it follows from Lemma 3, (2.6) and (3.3) that, for fixed $t > 0$,

[12] For a similar argument see Bosanquet [4] Lemma 8.

$$\phi(t) = t^{2q}\sum_{n=1}^{\infty} n^q A_n(x^0)\gamma(t\sqrt{n})$$

$$= t^{2q}\lim_{y\to+0}\lim_{R\to\infty}\sum_{n=1}^{R} n^q A_n(x^0)e^{-ny}\gamma(t\sqrt{n})$$

$$= t^{2q}\lim_{y\to+0}\lim_{R\to\infty}\{-\int_1^{\sqrt{R}} uT(0,u)\left(\frac{1}{u}\frac{d}{du}\right)e^{-u^2y}\gamma(ut)\,du$$

$$(3.5) \qquad + T(0,\sqrt{R})e^{-Ry}\gamma(t\sqrt{R})\}$$

$$= t^{2q}\lim_{y\to+0}\lim_{R\to\infty}\{\sum_{m=0}^{h+1}(-)^{m+1}[T(m,u)\left(\frac{1}{u}\frac{d}{du}\right)^m e^{-u^2y}\gamma(ut)]_{u=\sqrt{R}}$$

$$+ (-)^h\int_1^{\sqrt{R}} uT(h+1,u)\left(\frac{1}{u}\frac{d}{du}\right)^{h+2}e^{-u^2y}\gamma(ut)\,du\}$$

$$= (-)^h t^{2q}\lim_{y\to+0}\int_1^{\infty} uT(h+1,u)\left(\frac{1}{u}\frac{d}{du}\right)^{h+2}e^{-u^2y}\gamma(ut)\,du$$

$$= (-)^h t^{2q}\lim_{y\to+0}\sum_{\nu=0}^{h+2}\binom{h+2}{\nu}(-2y)^\nu\int_1^{\infty} uT(h+1,u)e^{-u^2y}\left(\frac{1}{u}\frac{d}{du}\right)^{h+2-\nu}\gamma(ut)\,du.$$

We now choose h to be the *greatest integer less than* τ. Then since[13] $T(h+1,u) = cu^{2h+2}\Delta_{x^0}{}^q S_u{}^{h+1}(f,x^0)$ it follows by consistency that $T(h+1,u) = o(u^{2h+2})$ and hence, using (2.6), that for $\nu = 1, 2, \cdots, h+2$ and fixed $t(>0)$,

$$\int_1^{\infty} uT(h+1,u)e^{-u^2y}\left(\frac{1}{u}\frac{d}{du}\right)^{h+2-\nu}\gamma(ut)\,du$$

$$= \int_1^{\infty} e^{-u^2y}o(u^{h-\mu-2q+\nu+\frac{1}{2}})\,du$$

$$= o(y^{-\nu}),$$

as $y\to+0$, if $p > \delta - \frac{1}{2}(k-3)$. Hence from (3.5), if

$$p > \max[\delta - \tfrac{1}{2}(k-3), \tfrac{1}{2}(k+1)],$$

$$(3.6) \qquad \phi(t) = (-)^h t^{2q}\lim_{y\to+0}\int_1^{\infty} uT(h+1,u)e^{-u^2y}\left(\frac{1}{u}\frac{d}{du}\right)^{h+2}\gamma(ut)\,du.$$

We now write, as in Lemma 2,

$$\lambda(u) = \sum_{\nu=q}^{\infty}(-)^\nu\frac{u^{2\nu-2q}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)}$$

$$P(u) = \sum_{\nu=0}^{q-1}(-)^\nu\frac{u^{2\nu-2q}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)},$$

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[13] Throughout what follows $c_0, c_1, c_2, \cdots$ denote constants not necessarily the same at each occurrence.

and

$$(\frac{1}{u}\frac{d}{du})^{h+2}P(u)=\sum_{\nu=0}^{q-1}b_\nu u^{+2\nu-2q-2h-4}$$

where

$$b_\nu=(-)^\nu\frac{(2\nu-2q)\cdots(2\nu-2q-2h-2)}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)} \tag{3.7}$$

Substituting the relation

$$\gamma(u)=P(u)+\lambda(u)$$

in (3.6) we obtain

$$\begin{aligned}\phi(t)&=\sum_{\nu=0}^{q-1}c_\nu t^{2\nu}+(-)^h t^{2q}\lim_{y\to+0}\int_1^\infty uT(h+1,u)e^{-u^2y}(\frac{1}{u}\frac{d}{du})^{h+2}\lambda(ut)\,du\\&=\sum_{\nu=0}^{q-1}c_\nu t^{2\nu}+t^{2q}I(t)\end{aligned} \tag{3.8}$$

say, where for $\nu=0,1,\cdots,q-1$,

$$\begin{aligned}c_\nu&=(-)^h b_\nu\lim_{y\to+0}\int_1^\infty T(h+1,u)e^{-u^2y}u^{2\nu-2q-2h-3}\,du\\&=(-)^h b_\nu\int_1^\infty T(h+1,u)u^{2\nu-2q-2h-3}\,du\end{aligned} \tag{3.9}$$

since $(T(h+1,u)=O(u^{2h+2}))$ this last integral is absolutely convergent.

We first show that

$$I(t)=O(1), \tag{3.10}$$

as $t\to+0$.

If τ is an integer, then $\tau=h+1$ and so by Lemma 2,

$$\begin{aligned}I(t)&=(-)^{\tau-1}\lim_{y\to+0}\int_1^\infty uT(\tau,u)e^{-u^2y}(\frac{1}{u}\frac{d}{du})^{\tau+1}\lambda(ut)\,du\\&=(-)^{\tau-1}\int_1^\infty uT(\tau,u)(\frac{1}{u}\frac{d}{du})^{\tau+1}\lambda(ut)\,du\\&=O(t^{2\tau+2}\int_1^{t^{-1}}u^{2\tau+1}\,du)+O(\int_{t^{-1}}^\infty u^{2\tau+1}t^{2\tau+2}[(ut)^{-\mu-2q-\tau-\frac{3}{2}}+(ut)^{-2\tau-4}]du)\\&=O(1)\end{aligned}$$

as $t\to+0$ provided that $p>\delta-\frac{1}{2}(k-3)$.

If τ is not an integer, we write

$$T(h+1,u)=c\int_1^u(u^2-v^2)^{h-\tau}vT(\tau,v)\,dv$$

where c is independent of u. We then substitute this expression into the integral defining $I(t)$ and invert the order of integration. Estimating the inner integral by standard methods then leads to (3.10).[14]

Collecting results we have, from (3.8),

$$\phi(t)=\sum_{\nu=0}^{q-1} c_\nu t^{2\nu}+O(t^{2q})$$

i.e., using (3.4), that

$$f_{x^0,p}(t)=\Gamma(\mu+1)\sum_{\nu=0}^{q-1}\frac{a_\nu t^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)}+O(t^{2q})$$

where

$$a_\nu=c_\nu 2^{2\nu}\nu!\Gamma(\nu+\mu+1) \qquad (\nu=1,\cdots,q-1)$$

$$a_0=c_0+A_0/\Gamma(\mu+1).$$

Now

$$g_{p,k}(t)=\frac{2t^{-2p-k+2}}{B(p,\tfrac{1}{2}k)}\int_0^t (t^2-u^2)^{p-1}u^{k-1}g(u)\,du$$

so that

$$\frac{\Gamma(\tfrac{1}{2}k)}{2^{2q}q!\Gamma(q+\tfrac{1}{2}k)}g_{p,k}(t)$$

$$=\frac{2t^{-2p-k+2}\Gamma(p+\tfrac{1}{2}k)}{\Gamma(\tfrac{1}{2}k)\Gamma(p)}\int_0^t (t^2-u^2)^{p-1}u^{k-1}\{f_{x^0}(u)-\Gamma(\tfrac{1}{2}k)\sum_{\nu=0}^{q-1}\frac{a_\nu u^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu+\tfrac{1}{2}k)}\}du$$

$$=f_{x^0,p}(t)-\Gamma(\mu+1)\sum_{\nu=0}^{q-1}\frac{a_\nu t^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu+\mu+1)}$$

$$=O(t^{2q})$$

as $t\to+0$.

We have thus shown that if $(-)^q\sum_{n=1}^{\infty} n^q A_n(x^0)$ is summable $(n,\delta+2q)$ to s then there exist constants a_ν such that (3.1) is satified for

$$p>\max[\delta-\tfrac{1}{2}(k-3),\tfrac{1}{2}(k+1)].$$

It remains to show that the a_ν, which are clearly unique, satisfy (3.2).

It follows from Lemma 1 that $(-)^{q-1}\sum_{n=0}^{\infty} n^{q-1}A_n(x^0)$ is summable $(n,\delta+2q-1)$, i.e. that

$$(-)^{q-1}\lim_{R\to\infty}\sum_{n\leqq R^2}(1-n/R^2)^{\delta+2q-1}n^{q-1}A_n(x^0)=s',$$

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[14] We omit the details, which are rather tedious. We shall only *use* Theorem 1 in the case when τ is an integer. The method outlined here is a combination of those in [5] page 215, Verblunsky [11], and [3].

say. From what we have already proved (with q replaced by $q-1$) there exist constants a_ν' such that

$$\frac{2^{2q-2}(q-1)!\Gamma(q-1+\frac{1}{2}k)}{\Gamma(\frac{1}{2}k)}\{f_{x^0}(t)-\Gamma(\tfrac{1}{2}k)\sum_{\nu=0}^{q-2}\frac{a_\nu' t^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu+\frac{1}{2}k)}\} \sim s't^{2q-2}(C_k)$$

as $t\to+0$. However we have just shown that

$$\frac{2^{2q}q!\Gamma(q+\frac{1}{2}k)}{\Gamma(\frac{1}{2}k)}\{f_{x^0}(t)-\Gamma(\tfrac{1}{2}k)\sum_{\nu=0}^{q-2}\frac{a_\nu t^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu+\frac{1}{2}k)}$$

$$=g(t)+a_{q-1}\frac{t^{2q-2}2^{2q}q!\Gamma(q+\frac{1}{2}k)}{2^{2q-2}(q-1)!\Gamma(q-1+\frac{1}{2}k)}$$

$$\sim st^{2q}(C_k)+a_{q-1}\frac{t^{2q-2}2^{2q}q!\Gamma(q+\frac{1}{2}k)}{2^{2q-2}(q-1)!\Gamma(q-1+\frac{1}{2}k)}$$

$$\sim a_{q-1}t^{2q-2}\frac{2^{2q}q!\Gamma(q+\frac{1}{2}k)}{2^{2q-2}(q-1)!\Gamma(q-1+\frac{1}{2}k)}(C_k)$$

as $t\to+0$. Hence $a_\nu=a_\nu'$ $(\nu=0,1,\cdots,q-2)$ and $a_{q-1}=s'$ i.e.

$$a_{q-1}=(-)^{q-1}\lim_{R\to\infty}\sum_{n\leqq R^2}(1-n/R^2)^{\delta+2q-1}n^{q-1}A_n(x^0).$$

Repetition of this argument shows that all the a_ν satisfy (3.2).

We next obtain a theorem which is converse to Theorem 1.

THEOREM 2. *If constants a_ν exist such that*

$$g(t)\sim st^{2q}(C_k,p)\quad(p\geqq 0) \tag{3.11}$$

as $t\to+0$ then $(-)^q\sum_{n=1}^{\infty}n^qA_n(x^0)$ is summable $(n,\delta+2q)$ to s whenever $\delta>p+\frac{1}{2}(k-1)$.

Moreover the constants a_ν are given by (3.2).

We observe first that we may suppose, without loss of generality that $s=a_\nu=0$ $(\nu=0,1,\cdots,q-1)$. For in the case

$$f(x)=P(x)=\sum_{\nu=0}^{q}a_\nu\{\frac{(x_1{}^0-x_1)^{2\nu}+\cdots+(x_k{}^0-x_k)^{2\nu}}{(2\nu)!}\}$$
$$(x_i{}^0\leqq x_i\leqq x_i{}^0+2\pi;\ i=1,\cdots,k)$$

where $a_q=s$, it follows from a known theorem [15] that $(-)^\nu\sum_{n=1}^{\infty}n^\nu A_n(x^0)$ is summable $(n,\delta+2\nu)$ $(\delta>\frac{1}{2}(k-1)$ to a_ν at $x=x^0$ $(\nu=0,\cdots,q)$.

[15] [7] page 124, Theorem 4. 62.

Moreover in this case[16] $g(t) = st^{2q}$ $(0 < t < \pi)$ so that if $f(x) \equiv P(x)$ then for any $p \geqq 0$ (3.11) is satisfied, the conclusion of Theorem 2 is satisfied and the a_ν satisfy (3.2) so that by considering $f(x) - P(x)$ in place of $f(x)$ we may suppose $s = 0 = a_\nu$ $(\nu = 0, \cdots, q-1)$.

We may thus suppose that

$$f_{x^0,p}(t) = o(t^{2q}) \tag{3.12}$$

as $t \to +0$.

Next[17]

$$\Delta_{x^0}{}^q S_R{}^{\delta+2q}(f, x^0) = R^{2q} \sum_{\nu=0}^{q} c_\nu S_R{}^{\delta+2q+\nu}(f, x^0) \tag{3.13}$$

where the c_ν $(\nu = 0, \cdots, q)$ are independent of R. Hence it is sufficient to show that

$$S_R{}^{\delta+2q+\nu}(f, x^0) = O(R^{-2q}) \qquad (\nu = 0, \cdots, q).$$

From a known result[18]

$$S_R{}^{\delta+2q+\nu}(f, x^0) = cR^{2p+k} \int_0^\infty f_{x^0,p}(t)\, t^{k+2p-1} V_{\delta+2q+\nu+p+\frac{1}{2}k}(tR)\, dt \tag{3.14}$$

provided that $\delta > p + \frac{1}{2}(k-1)$. Consequently, from (3.12) and (2.1),

$$\begin{aligned} R^{2q} S_R{}^{\delta+2q+\nu}(f, x^0) &= o(R^{2p+k+2q} \int_0^{R^{-1}} t^{k+2p+2q-1}\, dt) \\ &\quad + o(\int_{R^{-1}}^\infty \frac{R^{2p+k+2q} t^{k+2q+2p-1}}{(tR)^{\delta+2q+\nu+\frac{1}{2}k+p+\frac{1}{2}}}\, dt) \\ &= O(1) \end{aligned}$$

as $R \to \infty$ provided that $\delta > p + \frac{1}{2}(k-1)$.

Finally it follows from Theorem 1 that, since $(-)^q \sum_{n=1}^\infty n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ to s, the a_ν are given by (3.2).

4. Throughout this section we suppose that $h(t)$ is an even periodic function with period 2π which coincides with $t^{k-1} g(t)$ in $0 < t < \pi$ and that

$$h(t) \sim \tfrac{1}{2} b_0 + \sum_{n=1}^\infty b_n \cos nt.$$

The main result of this section may then be formulated as follows.

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[16] It is easy to verify that in this case

$$f_{x^0}(t) = \Gamma(\tfrac{1}{2}k) \sum_{\nu=0}^{q} \frac{a_\nu t^{2\nu}}{2^{2\nu} \nu! \Gamma(\nu + \frac{1}{2}k)}.$$

[17] [9] page 227.

[18] [10] page 214.

THEOREM 3. *If $\delta > \frac{1}{2}(k-1)$ $(k > 1)$ and q is a positive integer then $(-)^q \sum_{n=1}^{\infty} n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ to s if and only if there exist constants a_ν $(\nu = 0, \cdots, q-1)$ such that* (i)[19] *$g(t) \sim st^{2q}(C_k)$ as $t \to +0$ and* (ii) *if k is odd $\sum_{n=1}^{\infty} n^{2q+k-1} b_n$ is summable $(C, \delta + 2q + \frac{1}{2}k - \frac{1}{2})$ to $(-)^{\frac{1}{2}(k-1)+q}(2q + k - 1)!s$ or* (ii)$'$ *if k is even*

$$n^{2q+k} b_n \to (-)^{\frac{1}{2}k+q}(2q + k - 1)!\,2s/\pi \;(C, \delta + 2q + \tfrac{1}{2}k + \tfrac{1}{2})$$

as $n \to \infty$.

The a_ν, when they exist are given by (3.2).

We require two further lemmas.

LEMMA 4. *If $\delta > \frac{1}{2}(k-1)$ then*

$$R^{k+2q} \int_{\pi}^{\infty} g(t) t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt = O(1)$$

as $R \to \infty$. The same is true if $g(t)$ is replaced by t^{2q}.

Since, by (2.3), as $t \to \infty$,

$$\begin{aligned} |g(t)| &\leqq c\{|f_{x^0}(t)| + \Gamma(\tfrac{1}{2}k) \sum_{\nu=0}^{q-1} \frac{|a_\nu|\, t^{2\nu}}{2^{2\nu}\nu!\Gamma(\nu + \frac{1}{2}k)}\} \\ &= O(1)(C_k, 1) + O(t^{2q-2}) \\ &= O(t^{2q-2})(C_k, 1) \end{aligned}$$

it follows, using (2.1) that

$$\begin{aligned} \left| \int_{\pi}^{\infty} g(t) t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt \right| &\leqq cR^{-\delta-2q-\frac{1}{2}k-\frac{1}{2}} \int_{\pi}^{\infty} |g(t)|\, t^{k-1} \frac{dt}{t^{\delta+2q+\frac{1}{2}k+\frac{1}{2}}} \\ &= R^{-\delta-2q-\frac{1}{2}k-\frac{1}{2}} \left\{ \left[\frac{O(t^{k+2q-2})}{t^{\delta+2q+\frac{1}{2}k+\frac{1}{2}}} \right]_{\pi}^{\infty} + \int_{\pi}^{\infty} \frac{O(t^{k+2q-2})\, dt}{t^{\delta+2q+\frac{1}{2}k+\frac{3}{2}}} \right\} \\ &= o(R^{-2q-k}) \end{aligned}$$

as $R \to \infty$, since $\delta > \frac{1}{2}(k-1)$.

Finally, by (2.1)

$$\begin{aligned} \left| \int_{\pi}^{\infty} t^{2q+k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt \right| &\leqq cR^{-\delta-2q-\frac{1}{2}k-\frac{1}{2}} \int_{\pi}^{\infty} t^{-\delta+\frac{1}{2}k-\frac{3}{2}}\, dt \\ &= O(R^{-2q-k}) \end{aligned}$$

if $\delta > \frac{1}{2}(k-1)$.

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[19] Equivalently $(-)^q \sum n^q A_n(x^0)$ is summable by Riesz means of some order to s.

LEMMA 5. *In order that* $(-)^q \sum n^q A_n(x^0)$ *be summable* $(n, \delta + 2q)$ $(\delta > \frac{1}{2}(k-1))$ *to* s *it is necessary and sufficient that there exist* a_ν *such that* (i) $g(t) \sim st^{2q}(C_k)$ *as* $t \to +0$ *and* (ii)

$$R^{2q+k} \int_0^\infty g(t)\, t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt \to \frac{s\Gamma(q+\frac{1}{2}k)}{2^{\delta-\frac{1}{2}k+1}\Gamma(\delta+q+1)}$$

as $R \to \infty$.

Necessity. Suppose that $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ to s. It follows from Theorem 1 that constants a_ν exist such that $g(t) \sim st^{2q}(C_k)$ as $t \to +0$. If

$$P(x) = \sum_{\nu=0}^{q} a_\nu \left\{ \frac{(x_1 - x_1{}^0)^{2\nu} + \cdots + (x_k - x_k{}^0)^{2\nu}}{(2\nu)!} \right\} (a_q = s)$$

$$(x_i{}^0 \leqq x_i \leqq x_i{}^0 + 2\pi;\ i = 1, \cdots, k)$$

then, as in the working for Theorem 2, when $f(x) = P(x)$ the a_ν satisfy (3.2) and $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ $(\delta > \frac{1}{2}(k-1))$ to s. Moreover, since $g(t) = st^{2q}$ $(0 < t < \pi)$ it follows that (i) is satisfied. Condition (ii) is satisfied since, using Lemma 4,

$$\begin{aligned} R^{k+2q} &\int_0^\infty g(t)\, t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt \\ &= R^{k+2q} \int_0^\pi g(t)\, t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt + o(1) \\ &= sR^{k+2q} \int_0^\pi t^{2q+k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt + o(1) \\ &= sR^{k+2q} \int_0^\infty t^{2q+k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt + o(1) \\ &= \frac{s\Gamma(q+\frac{1}{2}k)}{2^{\delta-\frac{1}{2}k+1}\Gamma(\delta+q+1)} + O(1). \end{aligned}$$ [20]

Hence by subtracting $P(x)$ from $f(x)$ we may suppose that $s = a_\nu = 0$ $(\nu = 0, \cdots, q-1)$, i.e. it is sufficient to prove that

$$R^{2q+k} \int_0^\infty f_{x^0}(t)\, t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt = o(1) \tag{4.1}$$

as $R \to \infty$.

We observe that, by Bochner's formula, (4.1) is equivalent to

$$R^{2q} S_R{}^{\delta+2q}(f, x^0) = o(1) \tag{4.2}$$

as $R \to \infty$.

[20] [11] page 994, Formula 10.

Since $f_{x^0}(t) = O(t^{2q})(C_k)$ as $t \to +0$ it follows, as in the working for Theorem 2 that, for some τ_0,

$$(4.3) \qquad R^{2q} S_R^{\tau+2q}(f, x^0) = O(1) \qquad (\tau \geqq \tau_0)$$

as $R \to \infty$. Further, by (3.13),

$$(4.4) \qquad \Delta_{x^0}{}^q S_R^{\tau+2q}(f, x^0) = \sum_{\nu=0}^{q} c_\nu R^{2q} S_R^{\tau+2q+\nu}(f, x^0).$$

Since, by hypothesis, the left hand side of (4.4) is $O(1)$ as $R \to \infty$ when $\tau = \delta$ it follows by successive applications of (4.3), (4.4) and the consistency of Riesz means that (4.2) is satisfied.

Sufficiency. We may suppose, as before that $a_\nu = s = 0$ $(\nu = 0, 1, \cdots, q-1)$ and that (4.1), or equivalently (4.2), holds. The result then follows, by consistency from (4.4) with τ replaced by δ.

Proof of Theorem 3.[21] Firstly, we may suppose that $s = 0$. For in the case when

$$f(x) = P(x) = s\left\{\frac{(x_1 - x_1{}^0)^{2q} + \cdots + (x_k - x_k{}^0)^{2q}}{(2q)!}\right\}$$

$$(x_i{}^0 \leqq x_i \leqq x_i{}^0 + 2\pi;\, i = 1, \cdots, k)$$

we have, by arguments similar to those used in Theorem 2, that $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ $(\delta > \frac{1}{2}(k-1)$ to s, and if we put $a_\nu = 0$ $(\nu = 0, 1, \cdots, q-1)$ that the a_ν satisfy (3.2). Moreover in this case, for $0 < t < \pi$, $g(t) = st^{2q}$. Hence, if k is odd, $(-)^{q+\frac{1}{2}(k-1)} \sum_{n=1}^{\infty} n^{2q+k-1} b_n$ is the $(2q + k - 1)$-th derived Fourier cosine series at $t = 0$ of st^{2q+k-1} and is therefore summable $(C, \delta + 2q + \frac{1}{2}k - \frac{1}{2})$ to $(2q + k - 1)!s$ whenever $\delta > \frac{1}{2}(k-1)$ ((14), p. 60). On the other hand if k is even a straightforward calculation of the Fourier cosine coefficients b_n of $s \mid t^{2q+k-1} \mid$ shows that $n^{2q+k} b_n \to (-)^{\frac{1}{2}k+q}(2q + k - 1)!2s/\pi$, $(C, \delta + 2q + \frac{1}{2}k + \frac{1}{2})$ $(\delta > \frac{1}{2}(k-1))$. Thus when $f(x) = P(x)$, $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ to s for any $\delta > \frac{1}{2}(k-1)$ and (i), (ii) or (ii)$'$ are satisfied. Hence if we consider $f(x) - P(x)$ in place of $f(x)$ we may suppose that $s = 0$.

Necessity. The case k odd. We suppose that k is odd, $\delta > \frac{1}{2}(k-1)$ and $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta + 2q)$ to 0. It follows from Lemma 5 that constants a_ν $(\nu = 0, 1, \cdots, q-1)$ (satisfying (3.2)) exist such that $g(t) = O(t^{2q})(C_k)$ as $t \to +0$, i.e. that (i) is satisfied, and that

$$R^{2q+k} \int_0^\infty g(t) t^{k-1} V_{\delta+2q+\frac{1}{2}k}(tR)\, dt = o(1)$$

[21] The proof of this theorem is similar to that of Theorem 1 of [13], we give brief details for completeness.

as $R \to \infty$, and hence by Lemma 4 that

$$R^{2q+k}\int_0^\pi g(t)\,t^{k-1}V_{\delta+2q+\frac{1}{2}k}(tR)\,dt = o(1) \tag{4.5}$$

as $R \to \infty$.

Now by (2.4), with $\beta = 2q + k - 1$ and δ replaced by $\delta + 2q + \frac{1}{2}k - \frac{1}{2}$, f by h we have

$$\begin{aligned} R^{2q+k-1}S_R^{\delta+2q+\frac{1}{2}k-\frac{1}{2}}(h,0) &= cR^{2q+k}\int_0^\pi h(t)\,V_{\delta+2q+\frac{1}{2}k}(tR)\,dt + o(1) \\ &= o(1), \end{aligned} \tag{4.6}$$

by (4.5).

But, since k is odd,

$$(1 - n^2/R^2)^{\delta+2q+\frac{1}{2}k-\frac{1}{2}}(n^2/R^2)^{q+\frac{1}{2}(k-1)} = \sum_{\nu=0}^{q+\frac{1}{2}(k-1)} c_\nu (1 - n^2/R^2)^{\delta+2q+\frac{1}{2}(k-1)+\nu},$$

so that

$$\begin{aligned} \sum_{n \leqq R}(1 - n^2/R^2)^{\delta+2q+\frac{1}{2}(k-1)} n^{2q+k-1} b_n &= \sum_{\nu=0}^{q+\frac{1}{2}(k-1)} c_\nu R^{2q+k-1} S_R^{\delta+2q+\frac{1}{2}(k-1)+\nu}(h,0) \\ &= o(1) \end{aligned} \tag{4.7}$$

as $R \to \infty$, by (4.6) and consistency; equivalently $\sum n^{2q+k-1}b_n$ is summable $(C, \delta + 2q + \frac{1}{2}k - \frac{1}{2})$ to zero.

The case k even. This is proved in a similar way (cf. (13)) and we omit the details.

Sufficiency. The case k odd. Suppose that there exist constants a_ν $(\nu = 0, \cdots, q-1)$ such that $g(t) = O(t^{2q})(C_k)$, and that $\sum n^{2q+k-1}b_n$ is summable $(C, \delta + 2q + \frac{1}{2}k - \frac{1}{2})$ to zero or equivalently that

$$\sum_{n<R}(1 - n^2/R^2)^{\delta+2q+\frac{1}{2}(k-1)} n^{2q+k-1} b_n = O(1).$$

It follows from (4.7) and consistency that either

$$R^{2q+k-1}S_R^{\delta+2q+\frac{1}{2}(k-1)}(h,0) = o(1) \tag{4.8}$$

as $R \to \infty$ or that for some sequence $\tau_1 < \tau_2 < \cdots < \tau_p \to \infty$,

$$R^{2q+k-1}S_R^{\tau_r+2q+\frac{1}{2}(k-1)}(h,0) \neq o(1) \qquad (r = 1, 2, \cdots) \tag{4.9}$$

as $R \to \infty$. However, since $g(t) = o(t^{2q})(C_k)$ as $t \to +0$ it follows from (4.6) (with δ replaced by τ), and arguments similar to those in the proof of Theorem 2, that for *some* τ_0 $R^{2q+k-1}S_R^{\tau+2q+\frac{1}{2}(k-1)}(h,0) = O(1)$ $(\tau \geqq \tau_0)$ i.e. that (4.9)

is false. Hence (4.8) holds and so by (4.6) and Lemmas 4 and 5 $\sum n^q A_n(x^0)$ is summable $(n, \delta+2q)$ to zero.

The case k even is proved in a similar way.

5. Supplementary remarks. Condition (ii) of Theorem 3 may be rephrased as "the $(2q+k-1)$-th derived Fourier series of $h(t)$ is summable $(C, \delta+2q+\frac{1}{2}k-\frac{1}{2})$ to $(2q+k-1)!s$ at $t=0$." Combining this observation with known results in the case $k=1$ ([3] Theorem 2, [13] Lemma 2) we obtain the following result:

THEOREM 4. *If $\delta > \frac{1}{2}(k-1)$, where k is an odd positive integer, then $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta+2q)$ to s if and only if there exist constants a_ν $(\nu = 0, \cdots, q-1)$ such that* (i) *$t^{-2q-k+1}h(t)$ is integrable in the Cesaro-Lebesgue sense in $(0, \pi)$ and* (ii) *its Fourier cosine series is summable $(C, \delta-\frac{1}{2}k+\frac{1}{2})$ to s at $t=0$.*

If they exist, the constants a_ν are given by (3.2).

For details of a similar calculation we refer to [13] Theorem 2.

It seems likely that the case k even of Theorem 3 can be reduced in a similar way, but the required one variable results are not known. On the other hand the imposition of additional conditions on $f(x)$ enables us to obtain various simplifications of Theorem 3 (case k even). We finally give a simple and useful result of this kind. Here we write $\omega(t)$ for the even periodic function which coincides with st^{2q+k-1} in $0 < t < \pi$. We also write

$$\text{(5.1)} \qquad \mathrm{X}(t) = h(t) - \omega(t)$$

and suppose

$$\text{(5.2)} \qquad \mathrm{X}(t) \sim \tfrac{1}{2}\gamma_0 + \sum \gamma_n \cos nt$$

$$\text{(5.3)} \qquad \omega(t) \sim \tfrac{1}{2}\delta_0 + \sum \delta_n \cos nt.$$

We then have:

THEOREM 5. *If $\delta > \frac{1}{2}(k-1)$ where k is an even integer and if there exist constants a_ν $(\nu = 0, \cdots, q-1)$ such that $t^{-2q-k}\mathrm{X}(t) \in L(0, \pi)$ then in order that $(-)^q \sum n^q A_n(x^0)$ be summable $(n, \delta+2q)$ to s it is necessary and sufficient* (i) *that the $(2q+k-1)$-th derived allied series of the even function $\mathrm{X}(t)$ be summable $(C, \delta+2q+\frac{1}{2}k-\frac{1}{2})$ at $t=0$ or equivalently* (ii) *that the allied series of the odd periodic function equal to $t^{-2q-k+1}\mathrm{X}(t)$ in $0 < t < \pi$, be summable $(C, \delta-\frac{1}{2}k+\frac{1}{2})$ at $t=0$.*

If they exist the constants a_ν are given by (3.2).

Proof. It is easily verified that $t^{-2q-k}\chi(t) \in L(0,\pi)$ implies that

$$\int_0^t \chi(u)\,du = o(t^{2q+k})$$

as $t \to +0$, and hence, from (5.1), that $g(t) \sim st^{2q}(C_k, 1)$ as $t \to +0$. Consequently the initial assumption of Theorem 5 ensures that condition (i) of Theorem 3 is satisfied. Moreover a straightforward calculation shows that for $\delta > \frac{1}{2}(k-1)$ $n^{2q+k}\delta_n \to (-)^{\frac{1}{2}k+q}(2q+k-1)!2s/\pi(C, \delta+2q+\frac{1}{2}k+\frac{1}{2})$ so that from (5.1), (5.2), (5.3)

$$n^{2q+k}b_n \to (-)^{\frac{1}{2}k+q}(2q+k-1)!2s/\pi(C, \delta+2q+\tfrac{1}{2}k+\tfrac{1}{2})$$

if and only if

$$n^{2q+k}\gamma_n = O(1)\,(C, \delta+2q+\tfrac{1}{2}k+\tfrac{1}{2}) \tag{5.4}$$

as $n \to \infty$ i.e. condition (ii)′ of Theorem 3 is satisfied if and only if (5.4) holds.

Sufficiency of (i). Suppose that $\sum_{n=1}^{\infty} n^{2q+k-1}\gamma_n$ is summable $(C, \delta+2q+\frac{1}{2}k-\frac{1}{2})$ then $n^{2q+k}\gamma_n = o(1)$ $(C, \delta+2q+\frac{1}{2}k+\frac{1}{2})$ i.e. (5.4) holds so that condition (ii)′ of Theorem 3 is satisfied. Since condition (i) of that theorem is also satisfied it follows that $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta+2q)$ to s.

Necessity of (i). Suppose that $(-)^q \sum n^q A_n(x^0)$ is summable $(n, \delta+2q)$ to s then condition (ii)′ of Theorem 3 is satisfied and hence (5.4) holds. Since $t^{-2q-k}\chi(t) \in L(0,\pi)$ it follows [3] that the $(2q+k-1)$-th derived allied series at $t=0$ of $\chi(t)$ is summable (C) i.e. $\sum n^{2q+k-1}\gamma_n$ is summable (C). This together with (5.4) implies that $\sum n^{2q+k-1}\gamma_n$ is summable $(C, \delta, +2q+\frac{1}{2}k-\frac{1}{2})$.

The equivalence of conditions (i) and (ii) follows, in virtue of the integrability of $t^{-2q-k}\chi(t)$, from a result due to Bosanquet [3].

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THE COHOMOLOGY OF DIFFERENTIABLE TRANSFORMATION GROUPS.* [1]

By Richard S. Palais and Thomas E. Stewart.

1. Introduction. If a Lie group G acts differentiably on a manifold $\mathfrak{M}$ then various spaces of tensor field on $\mathfrak{M}$ become in a natural way modules for the Lie algebra $\mathfrak{G}$ of G and the cohomology of $\mathfrak{G}$ with these coefficient modules in certain cases carries interesting information about the action of G. In this paper we will discuss this situation, at first in a somewhat more abstract setup, and develop a method for computing these cohomology groups in certain cases. In particular we shall show that if G is compact and semi-simple then even though these modules are infinite dimensional the conclusions of the First and Second Whitehead Lemmas [10], [11] are valid; namely the first and second cohomology groups are trivial. As one consequence we will show that differentiable actions of compact, semi-simple Lie groups admit only trivial infinitesimal deformations (§ 11) a fact whose global analogue will be found in [7]. Our second and motivating application of these general cohomology results is to a question initiated by one of the authors in [9]. Namely if a Lie group G acts differentiably on the base space of a differentiable fiber bundle B over M can G be made to act differentiably on B so as to be equivariant with respect to the fiber projection and so that each operation of G on B is a bundle map. We show here that the answer is yes if G is compact and simply connected and if the structural group of B is a solvable Lie group, and moreover that the way of "lifting" the action of G to B is essentially unique.

2. Cohomology and invariant cohomology of Lie algebras. In the following L is a finite dimensional Lie algebra over a field F of characteristic zero. For the defintion of an L-module and a complete discussion of the cohomology of L with coefficients in an L-module we refer the reader to [2] or [3] (we shall use the notation of the latter). The notion of the invariant cohomology of L with coefficients in an L-module is implicit in a number of

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papers, however the lack of an exposition with the relevant facts that we shall need makes the following discussion desirable.

Recall that if M is an L-module then the space $C^p(L,M)$ of p-cochains of L with coefficients in M is defined to be M if $p=0$ and to be the space of alternating multilinear maps of L^p into M if p is a positive integer. Then $C^*(L,M)$ the cochain complex of L with coefficients in M is the graded vector space $\bigoplus_{p\geqq 0} C^p(L,M)$ with the differential d of degree $+1$ defined by $dm(X)=X\cdot m$ for $m\in M$ and

$$dc(X_1,\cdots,X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i+1}X_i(c(X_1,\cdots,\hat{X}_i,\cdots,X_{p+1}))$$
$$(1)\qquad +\sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots,\hat{X}_j,\cdots,X_{p+1})$$

for $c\in C^p(L,M)$, $p>0$. We recall from [3] that $d^2=0$ so there are defined graded vector spaces $Z^*(L,M)$ and $B^*(L,M)$ of cocycles and coboundaries and their quotient $H^*(L,M)$ the cohomology space of L with coefficients in M. We recall also that $C^*(L,M)$ becomes a graded L-module with $M=C^0(L,M)$ as a submodule if we define

$$(Xc)(X_1,\cdots,X_p)=X(c(X_1,\cdots,X_p))-\sum_{i=1}^{q}c(X_1,\cdots,[X,X_i],\cdots X_p)$$
$$(2)\qquad =X(c(X_1,\cdots,X_p))-\sum_{i=1}^{q}(-1)^{i+1}c([X,X_i],X_1,\cdots,\hat{X}_i,\cdots X_p)$$

for $c\in C^p(L,M)$, $p>0$, and that

$$(3)\qquad d(Xc)=X(dc)$$

for $X\in L$, $c\in C^*(L,M)$. It follows from (1) and (2) that for $c\in C^p(L,M)$

$$dc(X_1,\cdots,X_{p+1})=\sum_{i=1}^{p+1}(-1)^{i+1}(X_ic)(X_1,\cdots,\hat{X}_i,\cdots X_{p+1})$$
$$+\sum_{i<j}(-1)^{i+j+1}c([X_i,X_j],X_1,\cdots,\hat{X}_i,\cdots\hat{X}_j,\cdots X_{p+1})$$

We define a cochain $c\in C^*(L,M)$ to be *invariant* if $Xc=0$ for all $X\in L$. It follows from (3) that the graded subspace $C_I^*(L,M)=\bigoplus_{p\geqq 0} C_I^p(L,M)$ of $C^*(L,M)$ consisting of invariant cochains is stable under d and so gives rise to the graded vector spaces $Z_I^*(L,M)$ and $B_I^*(L,M)$ of invariant cocycles and invariant coboundaries (N.B. $B_I^*(L,M)=dC_I^*(L,M)$ *not* the possibly larger $C_I^*(L,M)\cap B^*(L,M)$) and their quotient $H_I^*(L,M)$ which we call the *invariant cohomology* of L with coefficients in M. By the *natural homomorphism* of $H_I^*(L,M)$ into $H^*(L,M)$ we mean the homomorphism induced

by the inclusion of $C_I^*(L,M)$ in $C^*(L,M)$. We note from (4) that $d_I = d\,|\,C_I^*(L,M)$ is given on $C^p(L,M)$ by

$$(5)\quad d_I c(X_1,\cdots,X_{p+1}) = -\sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\cdots X_i,\cdots X_j,\cdots X_{p+1})$$

An L-module M is called *trivial* if $Xm = 0$ for all $X \in L$, $m \in M$ (i.e. if $C_I^0(L,M) = C^0(L,M) = M$). Any vector space V over F can be considered a trivial L-module, in which case we write d_T for the differential on $C^*(L,V)$. Clearly from (1)

$$(6)\quad d_T c(X_1,\cdots,X_p) = \sum_{i<j}(-1)^{i+j}c([X_i,X_j],X_1,\cdots X_i\cdots X_j,\cdots X_{p+1}).$$

In particular we regard F as a trivial L-module and write $C^*(L,F) = C^*(L)$ and $H^*(L,F) = H^*(L)$. Since $C^1(L) = L^*$, the dual space of L, and since by (6) $d_T c(X,Y) = c([X,Y])$ for $c \in C^1(L)$ it follows that $Z^1(L) = [L,L]^0$, the annihilator of $[L,L]$. Since $B^1(L)$ is clearly zero (for $df(X) = 0$ for $f \in F$) it follows that $H^1(1) \cong Z^1(L) = [L,L]^0 \cong (L/[L,L])^*$. Thus $H^1(L)$ is trivial if and only if L is its own commutator subalgebra. Note also that H^* is an additive functor, i.e. $H^*(L, M \oplus N) \cong H^*(L,M) \oplus H^*(L,N)$, so it follows that if $H^p(L) = 0$ then $H^p(L,V) = 0$ for any trivial finite dimensional L-module V.

If M is any L-module then for each $X \in L$ there is an endomorphism i_X of $C^*(L,M)$ homogeneous of degree -1 defined by $i_X m = 0$ for $m \in M = C^0(L,M)$ and $(i_X c)(X_1,\cdots,X_{p-1}) = c(X,X_1,\cdots,X_{p-1})$ for $c \in C^p(L,M)$, $p > 0$. For later reference we recall from [3] that the module operations of L on $C^*(L,M)$ is related to d and the operations i_X by

$$(7)\qquad Xc = di_X c + i_X dc \qquad X \in L,\ c \in C^*(L,M).$$

3. An extension of the Whitehead lemmas. We will show in § 8 that if $\mathfrak{G}$ is the Lie algebra of a compact Lie group then for a certain class of $\mathfrak{G}$-modules M the natural homomorphism of $H_I^*(\mathfrak{G},M)$ into $H^*(\mathfrak{G},M)$ is an isomorphism onto. If $\mathfrak{G}$ is semi-simple it follows from the theorems we are above to prove that for such $\mathfrak{G}$-modules $H^1(\mathfrak{G},M)$ and $H^2(\mathfrak{G},M)$ are trivial, a fact that would also be a consequence of the Whitehead Lemmas ([10] and [11]) if M were finite dimensional. It is in this sense that this theorem extends the Whitehead lemmas.

THEOREM. *If L is a finite dimensional Lie algebra such that $H^1(L) = 0$ then $H_I^1(L,M) = 0$ for all L-modules M. If in addition $H^2(L) = 0$ then $H_I^2(L,M) = 0$ for all L-modules M.*

Proof. $C_I^0(L,M)=\{m\in M\,|\,Xm=0$ for all $X\in L\}$ so if $m\in C_I^0(L,M)$ then $dm(X)=Xm=0$ for all $X\in L$ hence $B_I^1(L,M)=0$ and $H_I^1(L,M)\cong Z_I^1(L,M)$. By (5) of § 2 if $c\in C_I^1(L,M)$ then $dc(X,Y)=c([X,Y])$ so $Z_I^1(L,M)=\{c\in C_I^1(L,M)\,|\,c$ is zero on $[L,L]\}$. But $H^1(L)=0$ is equivalent to $L=[L,L]$ and hence implies $H_I^1(L,M)\cong Z_I^1(L,M)=0$.

Now let $c\in Z_I^2(L,M)$ and let V be the finite dimensional subspace of M spanned by $\{c(X,Y)\,|\,X,Y\in L\}$. We consider V as a trivial L-module. Assuming $H^2(L)=0$ it follows that $H^2(L,V)=0$. Now $c\in C^2(L,V)$ and comparing (5) and (6) of § 2 we see that $d_Tc=-\,d_Ic=0$ so $c\in Z^2(L,V)=B^2(L,V)$ and hence $C=d_T\theta$ for some $\theta\in C^1(L,V)\subset C^1(L,M)$. It will suffice to prove that $\theta\in C_I^1(L,M)$, for then comparing (5) and (6) of § 2 again $d_I(-\theta)=-\,d_I\theta=d_T\theta=c$ so $c\in B_I^2(L,M)$ and we will have shown $Z_I^2(L,M)=B_I^2(L,M)$. Now if $X,X_1,X_2\in L$ then from (2) of § 2

$$\begin{aligned}0=(Xc)(X_1,X_2)&=X(c(X_1,X_2))-c([X,X_1],X_2)-c(X_1,[X,X_2])\\&=X(\theta([X_1,X_2]))-\theta([[X,X_1],X_2])-\theta([X_1,[X,X_2]])\end{aligned}$$

and by the Jacobi identity and the linearity of θ it follows that

$$0=X(\theta([X_1,X_2]))-\theta([X,[X_1,X_2]])$$

so, referring to (2) of § 2 again, $(X\theta)([X_1,X_2])=0$. Thus $X\theta$ vanishes on $[L,L]=L$ so θ is invariant. q. e. d.

4. Topological G-Modules. In this and succeeding sections G will denote a compact Lie group, G_0 its identity component, and $\mathfrak{G}$ its Lie algebra of left invariant vector fields. By a *topological G-module* we shall mean a complete, metrizable, locally convex, real topological vector space (a Frechét space) M together with a fixed homomorphism T of G into the group of automorphisms of M such that for each $m\in M$ the map $g\to T(g)m$ of G into M is continuous. We denote the space of continuous linear functionals on M by M^* and we write $\langle m,l\rangle$ for $l(m)$ if $(l,m)\in M^*\times M$ (which of the many possible topologies to put on M^* is irrelevant for our purposes and we shall always regard M^* as untopologized). In general we will drop explicit reference to T and simply write gm instead of $T(g)m$. Note that since M is a Frechét space and for each $m\in M$ the orbit $\{gm\,|\,g\in G\}$ is compact, and hence bounded, it follows from the principle of uniform boundedness that

4.1. Theorem. *Given a neighborhood V of zero in the topological G-module M there is a neighborhood U of zero in M such that $gu\in V$ for all $(g,u)\in G\times U$. Equivalently if $\{m_r\}$ is a sequence in M converging to m then $\{gm_r\}$ converges to gm uniformly for $g\in G$.*

Noting that $g_r m_r - gm = g_r(m_r - m) + (g_r m - gm)$ it follows from 4.1 that if $g_r \to g$ then $g_r m_r \to gm$.

4.2. COROLLARY. *If M is a topological G module then $(g, m) \to gm$ is a continuous map of $G \times M$ into M.*

Now let M be a topological G-module and M^X the linear space of functions from a compact Hausdorff space X into M. We give M^X the topology of uniform convergence, i.e. a typical neighborhood of zero is $\{f \in M^X \mid f(X) \subseteq U\}$ where U is some neighborhood of zero in M. By a step function from X to M we mean an element $f \in M^X$ whose range is a finite subset $\{m_1, \cdots, m_r\}$ of M such that each $f^{-1}(m_i)$ is Borel measurable. Since a continuous function from X to M is uniformly continuous it follows easily that the space $C(X, M)$ of continuous maps of X to M is included in the closure of the space $S(X, M)$ of step functions. Given a Radon measure μ on X and $f \in S(X, M)$ we define $\int f d\mu = \sum_{m \in M} \mu(f^{-1}(m))m$. Then $f \to \int f d\mu$ is a continuous linear map from $S(X, M)$ into M and so (because M is complete) extends to a continuous linear map of the closure of $S(X, M)$ into M. Henceforth we shall only be concerned with $\int f d\mu$ when f is continuous and we shall need the following obvious facts.

(a) $f_\alpha \to f$ uniformly on $X \Rightarrow \int f_\alpha d\mu \to \int f d\mu$

(b) $\int f d\mu$ is bilinear in f and μ

(c) If $\mu(X) = 1$ then $\int f d\mu \in$ closed convex hull of $f(X)$.

(d) If T is a continuous linear map of M into a complete topological vector space then $\int (Tf) d\mu = T \int f d\mu$.

The space $C(G)$ of continuous real valued functions on G is a Banach space in the supremum norm and becomes a topological G-module under the operations given by $(gf)(x) = f(g^{-1}x)$. If $f \in C(G)$ we define $f * m$, for m an element of a topological G-module M, to be the integral of the continuous map $g \to f(g)gm$ of G into M with respect to normalized Haar measure on G. If $l \in M^*$ then by (d) above $\langle f * m, l\rangle = \int f(g)\langle gm, l\rangle d \cdot g$. From this and the invariance of Haar measure it follows that $\langle (gf) * m, l\rangle = \langle g(f * m), l\rangle$ and since M^* separates points of M it follows that $(gf) * m = g(f * m)$.

4.3. THEOREM. *If M is a topological G-module then*

(1) *$(f, m) \to f * m$ is a continuous bilinear map of $C(G) \times M$ into M*

(2) *If $f \in C(G)$ is positive and $\int f(g) dg = 1$ then $f * m \in$ closed convex hull of $\{gm \mid f(g) \neq 0\}$.*

(3) *If $m \in M$ then $f \to f * m$ is an equivariant continuous linear map of $C(G)$ into M and m is in the closure of its range.*

Proof. Statement (1) is immediate from (a) and (b) above and state- (2) follows from (c). The equivariance of $f \to f * m$ (i.e. the fact that $(gf) * m = g(f * m)$ was shown above so it remains only to show that if W is a closed convex neighborhood of m in M then there is an $f \in C(G)$ such that $f * m \in W$. Let U be a neighborhood of the identity in G such that $gm \in W$ for $g \in U$ and let f be a continuous positive function of integral one on G with support in U. Then by (2) $f * m \in W$. q. e. d.

5. Almost invariant vectors. An element m_0 of a topological G-module M is called *almost invariant* if $\{gm_0 | g \in G\}$ spans a finite dimensional subspace of M. We denote the set of almost invariant vectors in M by M_0. Clearly M_0 is a subspace of M invariant under G, and in fact M_0 is just the linear span of the finite limensional G-invariant subspaces of M. An almost invariant vector in $C(G)$ is called an almost invariant function on G. From the bilinearity of $(f, m) \to f * m$ and the equivariance of $f \to f * m$ it follows that if f_0 is an almost invariant function and m any element of a topological G-module M then $f_0 * m \in M_0$. Now the Peter-Weyl theorem [1, p. 203] says that the almost invariant functions are dense in $C(G)$. Since we known that m is adherent to $\{f * m | f \in C(G)\}$ and that $f \to f * m$ is continuous we have the following essentially known result.

5.1. THEOREM. *In any topological G-module M the subspace M_0 of almost invariant vectors is dense.*

5.2. THEOREM. *If M is any topological G-module then the subspace M_0 of almost invariant vectors has a $\mathfrak{G}$-module stducture defined by $Xm = \lim_{t \to 0} 1/t((\mathrm{Exp}\, tX)m - m)$ for $X \in \mathfrak{G}$, $m \in M_0$.*

Proof. If V is any finite dimensional invariant subspace of M_0 $(g, v) \to gv$ is an analytic map of $G \times V \to V$ (this follows for example from [1, Prop 1, p. 128]) so that the indicated limit exists for $m \in V$. That this gives a $\mathfrak{G}$-module on V follows from [1, Theorem 2, p. 113]. Since M_0 is the linear span of its finite dimensional invariant subspaces the theorem follows. q. e. d.

6. Differentiable *G*-modules.

6.1. LEMMA. *Let M be a topological G-module and $\{m_r\}$ a sequence in M converging to m. Given $X \in \mathfrak{G}$ and $l \in M^*$ suppose there exists a sequence*

$\{\omega_k\}$ *in* M *converging to* ω *such that* $d/dt\langle(\operatorname{Exp} tX)m_r, l\rangle = \langle(\operatorname{Exp} tX)\omega_r, l\rangle$. *Then* $d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\omega, l\rangle$.

Proof. By 4.1 gm_r converges to gm uniformly in g, so since l is linear and continuous and hence uniformly continuous, $\langle gm_r, l\rangle$ converges to $\langle gm, l\rangle$ uniformly in g and *a fortiori* $\langle(\operatorname{Exp} tX)m_k, l\rangle$ converges to $\langle(\operatorname{Exp} tX)m, l\rangle$ uniformly in t. Similarly $\langle(\operatorname{Exp} tX)\omega_k, l\rangle$ converges to $\langle(\operatorname{Exp} tX)\omega, l\rangle$ uniformly in t. By a theorem of elementary calculus this validates differentiation "under the limit sign" and gives the desired result. q. e. d.

6.2. THEOREM. *Let* M *be a topological G-module* $X \in \mathfrak{G}$, *and suppose there is a function* $\tilde{X}: M \to M$ *such that*

$$d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$$

for all $m \in M$ *and all* l *belonging to a separating family* $S \subset M^*$. *Then* $\tilde{X}$ *is a continuous linear map and for* $m \in M_0$ $Xm = \lim_{t\to 0} 1/t((\operatorname{Exp} TX)m - m)$.

Proof. The linearity of $\tilde{X}$ is immediate from the linearity of differentiation and the fact that S is separating. The preceding lemma gives immediately that $\tilde{X}$ has a closed graph and, since M is a Frechét space $\tilde{X}$ is continuous. If $m \in M_0$ then by 5.2 $1/t((\operatorname{Exp} tX)m - m)$ converges strongly and *a fortiori*, weakly to a limit say m'. By definition of $\tilde{X}$ $\langle m', l\rangle = \langle\tilde{X}m, l\rangle$ for $l \in S$ and again since S is separating $m' = \tilde{X}m$. q. e. d.

6.3. THEOREM. *Let* M *be a topological G-module,* $X \in \mathfrak{G}$, *and suppose that the linear map*

$$m \to \lim_{t\to 0} 1/t((\operatorname{Exp} tX)m - m)$$

of M_0 *into itself* (see 5.2) *is continuous. Then this map extends uniquely to a continuous linear map* $\tilde{X}$ *of* M *into itself and* $d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$ *for all* $m \in M$, $l \in M^*$.

Proof. The existence and uniqueness of $\tilde{X}$ follows from the completeness of M and the denseness of M_0 in M. Given $l \in M^*$, $m \in M_0$, and t_0 a real number $l_0: m \to \langle(\operatorname{Exp} t_0X)m, l\rangle$ is an element of M^* and

$$\left.\frac{d}{dt}\right|_{t=t_0}\langle(\operatorname{Exp} tX)m_0, l\rangle = \left.\frac{d}{dt}\right|_{t=0}\langle(\operatorname{Exp}(t+t_0)X)m_0, l\rangle$$

$$\left.\frac{d}{dt}\right|_{t=0}\langle(\operatorname{Exp} tX)m_0, l_0\rangle = \langle\tilde{X}m_0, l_0\rangle = \langle(\operatorname{Exp} t_0X)\tilde{X}m_0, l\rangle.$$

Thus $\langle(\operatorname{Exp} tX)m, l\rangle$ is differentiable for $m \in M_0$ and $l \in M^*$ and its derivative is $\langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$. If m is an arbitrary element of M choose a sequence $\{m_r\}$ in M_0 converging to m. Then $\tilde{X}m_r \to \tilde{X}m$ and for any $l \in M^*$

$d/dt\langle(\operatorname{Exp} tX)m_r, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m_r, l\rangle$ so by 6.1 $d/dt\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$. q. e. d.

6.4. THEOREM. *If M is a topological G-module then the following are equivalent conditions.*

(1) *For each $X \in \mathfrak{G}$ there is a function $\tilde{X}: M \to M$ such that for all l in a separating subset of M^* $\frac{d}{dt}\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)\tilde{X}m, l\rangle$ for all $m \in M$.*

(2) *For each $X \in \mathfrak{G}$ the map $X^0: m \to \lim_{t\to 0} 1/t((\operatorname{Exp} tX)m - m)$ of M_0 into itself* (see 5.2) *is continuous.*

(3) *For each $X \in \mathfrak{G}$ and $m \in M$ $t \to (\operatorname{Exp} tX)m$ has a weak derivative, X'_m, at $t = 0$.*

(4) *For each $X \in \mathfrak{G}$ there is a map $X^*: M \to M$ such that*

$$\frac{d}{dt}\langle(\operatorname{Exp} tX)m, l\rangle = \langle(\operatorname{Exp} tX)X^*m, l\rangle.$$

Moreover if any one, and hence all, of these conditions are satisfied then the maps $\tilde{X}$, X', and X^* are all equal and are the unique continuous linear map of M into intself which extends X^0.

Proof. It is clear that (4) implies (1) and that if $\tilde{X}$ and X^* exist that they are equal. From 6.2 it follows that (1) implies (2) and that if $\tilde{X}$ exists, it is the unique continuous linear extension of X^0. From 6.3 it follows that (2) implies (3) and that X^* is the unique continuous linear extension of X^0. It remains to show that (3) implies (4) and that $X' = X^*$. Assuming (3) holds let $l \in M^*$ and t_0 a real number and define $l_0 \in M^*$ by $m \to \langle(\operatorname{Exp} t_0X)m, l\rangle$. Then for any $m \in M$

$$\left.\frac{d}{dt}\right|_{t=t_0}\langle(\operatorname{Exp} tX)m, l\rangle = \left.\frac{d}{dt}\right|_{t=0}\langle(\operatorname{Exp}((t+t_0)X)m, l\rangle$$

$$\left.\frac{d}{dt}\right|_{t=0}\langle(\operatorname{Exp} tX)m, l_0\rangle = \langle X'm, l_0\rangle = \langle(\operatorname{Exp} t_0X)X'm, l\rangle.$$

This shows both that (4) holds and that $X^*m \equiv X'm$.

q. e. d.

6.5. *Definition.* A topological G-module is called differentiable if it satisfies any one and hence all of the properties (1)-(4) of 6.4.

Remark. It is easily shown by example that the maps $t \to (\operatorname{Exp} tX)m$ need not be strongly differentiable for all m in a differentiable G-module.

6.6. THEOREM. *If M is a differentiable G-module then M has a $\mathfrak{G}$-module structure which is characterized by the identity*

$$\langle Xm, l\rangle = \frac{d}{dt}\bigg|_{t=0} \langle (\operatorname{Exp} tX)m, l\rangle$$

for $X \in \mathfrak{G}$, $m \in M$, $l \in M^*$. Moreover each of the module operations of $\mathfrak{G}$ on M is continuous.

Proof. Immediate from 5.1, 5.2, and 6.4.

6.7. *Definition.* If M is a differentiable G-module then the $\mathfrak{G}$-module structure for M described in 6.6 is called the derived $\mathfrak{G}$-module.

Henceforth differentiable G-modules will be regarded without explicit mention as $\mathfrak{G}$-modules, the derived $\mathfrak{G}$-module always being understood.

6.8. THEOREM. *If M is a differentiable G-module then:*

(1) $\dfrac{d}{dt}\langle (\operatorname{Exp} tX)m, l\rangle = \langle (\operatorname{Exp} tX)Xm, l\rangle X \in \mathfrak{G},\ m \in M,\ l \in M^*$

(2) $gXg^{-1}m = (ad(g)X)m,\ X \in \mathfrak{G},\ g \in G,\ m \in M$

(3) *If $m \in M$ then $Xm = 0$ for all $X \in \mathfrak{G}$ if and only if $gm = m$ for all $g \in G_0$.*

Proof. (1) follows easily from 6.4 and the definition of the derived $\mathfrak{G}$ module structure. To prove (2) we note that if $l \in M^*$ and we write $\bar{l} = l \circ g$ then

$$\begin{aligned} \langle (ad(g)X)m, l\rangle &= \frac{d}{dt}\bigg|_{t=0} \langle \operatorname{Exp} t(ad(g)X)m, l\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle g(\operatorname{Exp} tX)g^{-1}m, l\rangle \\ &= \frac{d}{dt}\bigg|_{t=0} \langle (\operatorname{Exp} tX)(g^{-1}m), \bar{l}\rangle \\ &= \langle Xg^{-1}m, \bar{l}\rangle = \langle gXg^{-1}m, l\rangle. \end{aligned}$$

If $gm = m$ for all $g \in G_0$ then $(\operatorname{Exp} tX)m = m$ for all $X \in \mathfrak{G}$ and all real t hence for each $X \in \mathfrak{G}$ $t \to (\operatorname{Exp} tX)$ has strong and therefore weak derivative zero at $t = 0$ and by Definition $Xm = 0$. Conversely if $Xm = 0$ for a given $X \in \mathfrak{G}$ then by (1) $\dfrac{d}{dt}\langle \operatorname{Exp} tX, l\rangle \equiv 0$ so $\langle (\operatorname{Exp} tX)m, l\rangle \equiv \langle (\operatorname{Exp} 0X)m, l\rangle = \langle m, l\rangle$, $(\operatorname{Exp} tX)m \equiv m$. Since G is compact every element of G_0 is of the form $\operatorname{Exp} X$ and (3) follows. q. e. d.

7. The differentiable structure of $C^*(\mathfrak{G}, M)$. In this section we will show that if M is a differentiable G-module then $C^*(\mathfrak{G}, M)$ is in a natural way a differentiable G-module also. Moreover we shall show that the derived $\mathfrak{G}$-module structure for $C^*(\mathfrak{G}, M)$ coincides with the other "natural" $\mathfrak{G}$-module structure that it has *qua* cochain complex of the $\mathfrak{G}$-module M (§ 1, equation (2)). Finally we shall show that the differential d is continuous on $C^*(\mathfrak{G}, M)$ and commutes with each operation of G.

Since $C^*(\mathfrak{G}, M) = \bigoplus_p C^p(\mathfrak{G}, M)$ and $C^p(\mathfrak{G}, M) = 0$ for $p > \dim \mathfrak{G}$, to define a differentiable G-module structure on $C^*(\mathfrak{G}, M)$ it suffices to define one on each $C^p(\mathfrak{G}, M)$, $p > 0$. For a typical neighborhood of zero in $C^p(\mathfrak{G}, M)$ we take $\{c \in C^p(\mathfrak{G}, M) \mid c(X_1, \cdots, X_p) \in U \text{ if } X_1, \cdots, X_p \in B\}$ where B is a compact subset of $\mathfrak{G}$ and U a neighborhood of zero in M. Thus $c_\alpha \to c$ means that for each compact subset B of M c_α converges uniformly to c on B^p. The metrizeability, completeness, and local convexity of $C^p(\mathfrak{G}, M)$ follow directly from the corresponding properties of M. The operations of G on $C^p(\mathfrak{G}, M)$ are defined by $(gc)(X_1, \cdots, X_p) = g(c(ad(g^{-1})X_1, \cdots, ad(g^{-1})X_p))$. It is obvious that each such operation is an automorphism of $C^p(\mathfrak{G}, M)$ and that $(g_1 g_2)c = g_1(g_2 c)$. That $g \to gc$ is a continuous map of G into $C^p(\mathfrak{G}, M)$ for any $c \in C^p(\mathfrak{G}, M)$ is a straightforward argument which we leave to the reader. Given $X_1, \cdots, X_p$ in $\mathfrak{G}$ and a $l \in M^* c \to \langle c(X_1, \cdots, X_p), l \rangle$ is an element of $C^p(\mathfrak{G}, M))^*$ and the collection of such continuous linear functionals on $C^p(\mathfrak{G}, M)$ is clearly separating. Now put

$$g(t) = c(ad(\operatorname{Exp} - tX)X_1, \cdots, ad(\operatorname{Exp} - tX)X_p).$$

From the well-known fact that

$$\lim_{t \to t_0} \frac{1}{t - t_0} (ad(\operatorname{Exp} tX)Y - ad(\operatorname{Exp} t_0 X)Y) = ad(\operatorname{Exp} t_0 X)[X, Y]$$

it follows from the multilinearity of c that $g(t)$ is strongly differentiable at t_0 and that

$$g'(t_0) = -\sum_{i=1}^{p} c(ad(\operatorname{Exp} - t_0 X)X_1, \cdots, ad(\operatorname{Exp} - t_0 X)[X, X_i], \cdots, ad(\operatorname{Exp} - t_0 X)X_p).$$

(We have used the fact that a multilinear map of a finite dimensional space into a topological vector space is automatically continuous.) Putting $\lambda(t) = (\operatorname{Exp} tX)g(t)$ we have

$$\frac{1}{1 - t_0}(\lambda(t) - \lambda(t)) = \frac{1}{t - t_0}((\operatorname{Exp} tX)g(t_0) - g(t_0)) + (\operatorname{Exp} tX)\left(\frac{1}{t - t_0}(g(t) - g(t_0))\right)$$

By the joint continuity of $G \times C^p(\mathfrak{G}, M) \to C^p(\mathfrak{G}, M)$ the second term converges to $(\operatorname{Exp} t_0X)g'(t_0)$ as t converges to t_0 while by (1) of 6.8 the first term converges weakly to $(\operatorname{Exp} t_0X)Xg(0)$. Thus if $l \in M^*$ then

$$\frac{d}{dt}\Big|_{t=t_0} \langle ((\operatorname{Exp} tX)c)(X_1, \cdots, X_p), l\rangle = \frac{d}{dt}\Big|_{t=t_0} \langle \lambda(t), l\rangle$$
$$= \langle ((\operatorname{Exp} t_0X)(Xc))(X_1, \cdots, X_p), l\rangle$$

where $Xc \in C^p(\mathfrak{G}, M)$ is defined in (2) of § 1. It follows that condition (1) of 6.2 is satisfied by the topological G-module $C^p(\mathfrak{G}, M)$ and hence that $C^p(\mathfrak{G}, M)$ is a differentiable G-module, the derived $\mathfrak{G}$-module structure moreover being that given in (2) of § 1. If $\{c_n\}$ is a sequence in $C^p(\mathfrak{G}, M)$ converging to c then it is clear from (1) of § 1 that for any $X_1, \cdots, X_{p+1}$ $dc_\alpha(X_1, \cdots, X_{p+1})$ converges in M to $dc(X_1, \cdots, X_p)$, from which it is clear that the map d: $C^p(\mathfrak{G}, M) \to C^{p+1}(\mathfrak{G}, M)$ has a closed graph and so, the domain and range being Frechét spaces, d is continuous. A similar argument shows that for each $X \in \mathfrak{G}$ the linear map i_X: $C^p(\mathfrak{G}, M) \to C^{p-1}(\mathfrak{G}, M)$ is continuous.

From (2) of (6.8) it follows that $Xgc = g(ad(g^{-1})X)$ for all $g \in G$, $X \in \mathfrak{G}$ and $c \in C^*(\mathfrak{G}, M)$. Using this and equation (1) of § 1 a straightforward computation gives $g(dc) = d(gc)$.

8. The main theorem.

8.1. Lemma. *If M is a differentiable G-module then each operation of G_0 on $C^*(\mathfrak{G}, M)$ is chain homotopic to the identity, i.e. for each $g \in G_0$ there is a linear map $\lambda(g)$ of $C^*(\mathfrak{G}, M)$ into itself such $gc - c = d\lambda(g)c + \lambda(g)dc$ for all $c \in C^*(\mathfrak{G}, M)$. Moreover, the map $g \to \lambda(g)$ can be cosen so that $g \to \lambda(g)c$ is continuous for each $c \in C^*(\mathfrak{G}, M)$.*

Proof. Since G_0 is compact and connected we can find, for each $g \in G_0$, $X \in \mathfrak{G}$ such that $g = \exp X$. We define $\lambda(g) \cdot c = \int_0^1 (\operatorname{Exp} tX) i_X c \cdot dt$. Since $C^*(\mathfrak{G}, M)$ is a differentiable G-module from 6.8, (1) for any $l \in (C^*(\mathfrak{G}, M))^*$

$$\langle gc - c, l\rangle = \int_0^1 \frac{d}{dt} \langle (\operatorname{Exp} tX)c, l\rangle dt$$
$$= \int_0^1 \langle (\operatorname{Exp} tX)Xc, l\rangle dt$$
$$= \langle \int_0^1 ((\operatorname{Exp} tX)Xc\, dt), l\rangle$$

and recalling that $Xc = di_X c + i_X dc$ (§ 1, (7))

$$\langle gc - c, l\rangle = \langle \int_0^1 (\operatorname{Exp} tX)\, di_X c + \lambda(g)\, dc, l\rangle.$$

Since d commutes with $\operatorname{Exp} tX$ and integration we obtain

$$\langle gc - c, l\rangle = \langle d\lambda(g)c + \lambda(g)\, dc, l\rangle$$

This equation holds for all $l \in (C^*(\mathfrak{G}, M))^*$ and hence $\lambda(g)$ is a chain homotopy.

Now suppose that U is a small enough neighborhood of the identity so that the exponential map has a continuous inverse, f. Define then

$$\lambda_0(g)c = \int_0^1 (\operatorname{Exp} tf(g))\, i_{f(g)} c\, dt.$$

$\lambda_0(g)$ has the required properties and $g \to \lambda_0(g)c$ is continuous on U. Now choose $g_1, \cdots, g_n$ in G_0 so that $g_i U$ cover G_0, and choose $\lambda(g_i)$ satisfying the lemma. It is easily seen that if we define $\lambda_i(g)$ for $g = g_i u \in g_i U$ by

$$\lambda_i(g) = \lambda(g_i) + \lambda_0(u) + \lambda(g_i)\, d\lambda_0(u) + d\lambda(g_0)\lambda_0(u)$$

then $gc - c = (d\lambda_i(g) + \lambda_i(g)w)c$ and $g \to \lambda_i(g)c$ is continuous on $g_i U$ for each $c \in C^*(\mathfrak{G}, M)$. If we put $\phi_i(g)\lambda_i(g)c = 0$, $\{\phi_i\}$ a partition of unity refining the covering $\{g_i U\}$, for $g \notin g_i U$ then clearly $g \to \phi_i(g)\lambda_i(g)c$ is continuous on G_0, so if we put $\lambda(g)c = \sum_{i=1}^{n} \phi_i(g)\lambda_i(g)c$ then $g \to \lambda(g)c$ is continuous on U, and satisfies the requirements of the lemma.

8.2. LEMMA. *Let μ_0 be the normalized Haar measure on G_0. If M is a differentiable G-module, define $A: C^*(\mathfrak{G}, M) \to C^*(\mathfrak{G}, M)$ by $Ac = \int gc\, d\mu_0(g)$. Then there is a linear map λ of $C^*(\mathfrak{G}, M)$ into itself such that $Ac - c = d\lambda c + \lambda dc$ for all $c \in C^*(\mathfrak{G}, M)$.*

Proof. Define $\lambda c = \int (g)c\, d\mu_0(g)$ is chosen as in Lemma 8.1.

8.3. LEMMA. *Let M be a differentiable G-module and let $A: C^*(\mathfrak{G}, M) \to C^*(\mathfrak{G}, M)$ be the linear operator defined in Lemma 8.2. Then A has the following properties:*

(1) *It is a projection of $C^*(\mathfrak{G}, M)$ on $C_I^*(\mathfrak{G}, M)$.*

(2) *It commutes with d.*

(3) *If $z \in Z^*(\mathfrak{G}, M)$ then $Az - z \in B^*(\mathfrak{G}, M)$ i.e. Az is cohomologous to z.*

Proof. Property (1) follows directly from the invariance of Haar measure and statement (3) of 6.8 (applied to $C^*(\mathfrak{G}, M)$ instead of M). Statement (2) follows from the fact that d commutes with the operations of G on $C^*(\mathfrak{G}, M)$ and with the integral. Finally (3) is an immediate consequence of Lemma 8.2. q. e. d.

8.4. THE MAIN THEOREM. *If M is a derived $\mathfrak{G}$-module then the natural homomorphism $i^*: H_I^*(\mathfrak{G}, M) \to H^*(\mathfrak{G}, M)$ induced by the inclusion map of $C_I^*(\mathfrak{G}, M)$ in $C^*(\mathfrak{G}, M)$ is an isomorphism onto. In other words every cohomology class of $H^*(\mathfrak{G}, M)$ contains an invariant cocycle, and two invariant cocycles which differ by a coboundary differ by the coboundary of an invariant cochain.*

Proof. Immediate from 8.3.

8.5. COROLLARY. *If $\mathfrak{G}$ is semi-simple then $H^1(\mathfrak{G}, M) = H^2(\mathfrak{G}, M) = 0$ for all derived $\mathfrak{G}$-modules M.*

Proof. Immediate from 8.4 and the theorem of § 3.

9. The differentiability of tensor modules. Let $\mathfrak{M}$ be a differentiable G-space, i. e. $\mathfrak{M}$ is a differentiable $(= C^\infty)$ manifold and there is given a differentiable map $(g, p) \to gp$ of $G \times \mathfrak{M}$ into $\mathfrak{M}$ (the *action* of G on $\mathfrak{M}$) such that $ep \equiv p$ and $(gg^1)p \equiv g(g^1p)$. For each $X \in \mathfrak{G}$ there is a differentiable vector field X^* on $\mathfrak{M}$ defined by X^*_p is the tangent to $t \to (\mathrm{Exp} - tX)p$ at $t = 0$. The map $X \to X^*$ is a homomorphism of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on $\mathfrak{M}$ which is called the *infinitesimal generator* of the action of G on $\mathfrak{M}$. Let $\mathfrak{T}$ be the space of all differentiable tensor fields on $\mathfrak{M}$ of any fixed (mixed) rank and symmetry type with the usual "C^∞-topology" (i. e. convergence means uniform convergence of each component and of each partial derivative of any order of a component on any compact subset of a coordinate neighborhood). It is well-known that $\mathfrak{T}$ is a complete locally convex space and in fact a Montel space. Moreover if $\mathfrak{M}$ is second countable, as we henceforth assume, then $\mathfrak{T}$ is metrizable. Each diffeomorphism ϕ of $\mathfrak{M}$ induces an automorphism of $\mathfrak{T}$ in a well-known way and we will write this automorphism as ϕ also. Moreover if $\{\phi_n\}$ is a sequence of diffeomorphism of $\mathfrak{M}$ converging to a diffeomorphism ϕ in the C^∞-topology then it is clear that $\phi_n T \to \phi T$ for any $T \in \mathfrak{T}$. By a theorem of Montgomery [4] if we write g^* for the operation $p \to gp$ of an element of G on $\mathfrak{M}$ then $g \to g^*$ is continuous from G into the group of diffeomorphisms of $\mathfrak{M}$ with the C^∞-topology. It follows from these last two facts that

$\mathfrak{T}$ is a topological G-module if we define $gT = g^*T$. We shall call such a topological G-module a *module* of tensor fields (associated with the differentiable G-space $\mathfrak{M}$) and shall as usual write gT instead of g^*T. It follows easily from Theorem II of [5] that a module of tensor fields $\mathfrak{T}$ is always differentiable and that for $X \in \mathfrak{G}$ and $T \in \mathfrak{T}$ XT is just the usual Lie derivative of T with respect to the vector field X^*. We can now forget about the topology on $\mathfrak{T}$ and even the action of G on $\mathfrak{T}$. All that is important for the applications we shall make is summed up in

THEOREM. *Let $\mathfrak{M}$ be a differentiable G-space and $X \to X^*$ the infinitesimal generator of the action of G on $\mathfrak{M}$. Let $\mathfrak{T}$ be the space of all differentiable tensor fields on $\mathfrak{M}$ of a fixed rank and symmetry type, and for $X \in \mathfrak{G}$ and $T \in \mathfrak{T}$ let XT be the Lie derivative of T with respect to X^*. Then this makes $\mathfrak{T}$ into a derived $\mathfrak{G}$-module so that* (8.4 and 8.5) $H^*(\mathfrak{G}, \mathfrak{T}) \cong H_I^*(\mathfrak{G}, \mathfrak{T})$ *and if $\mathfrak{G}$ is semi-simple* $H^1(\mathfrak{G}, \mathfrak{T}) = H^2(\mathfrak{G}, \mathfrak{T}) = 0$.

10. $H^*(G, R) \cong H^*(\mathfrak{G})$ as a special case. Consider G as a differentiable G-space under the map $(g, p) \to pg^{-1}$. Then for each $X \in \mathfrak{T}$ the associated vector field X^* on G is just X itself. Let $\mathfrak{T}$ be the tensor module of all differentiable real valued functions on G. If ω is a p-form on G then ω defines an element c of $C^p(\mathfrak{G}, \mathfrak{T})$ by $c(X_1, \cdots, X_p)(x) = \omega_x((X_1)_x, \cdots, (X_p)_x)$. Conversely given $c \in C^p(\mathfrak{G}, \mathfrak{T})$ define a p-form ω on G as follows: given $x \in G$ and tangent vectors $Y_1, \cdots, Y_p$ at x let $\omega_x(Y_1, \cdots, Y_p) = c(X_1, \cdots, X_p)(x)$ where X_i is the element of $\mathfrak{G}$ satisfying $(X_i)_x = Y_i$. It is readily verified that ω is differentiable and that these two maps are mutually inverse linear isomorphisms between $C^*(\mathfrak{G}, \mathfrak{T})$ and the space $\Omega_0(G)$ of differentiable forms on G. Moreover it is a well-known fact (a proof can be found in [15]) that under this correspondence the differential on $C^*(\mathfrak{G}, \mathfrak{T})$ corresponds to the exterior derivative on $\Omega^*(G)$. Thus by deRham's Theorem $H^*(G, R) \cong H^*(\mathfrak{G}, \mathfrak{T})$ and so by the theorem of § 9 $H^*(G, R) \cong H_I^*(\mathfrak{G}, \mathfrak{T})$. For $c \in C^p(\mathfrak{G}, \mathfrak{T})$ we have

$$(gc)(X_1, \cdots, X_p)(x) = c(ad(g^{-1})X_1, \cdots, ad(g^{-1})X_p)(xg)$$

and if $X \in \mathfrak{G}$ then by left invariance $(ad(g^{-1})X)_{xg} = dRg(X_x)$ (where Rg is right translation by g). It follows that $c \in C_I^*(\mathfrak{G}, \mathfrak{T})$ if and only if the corresponding form ω is right invariant. This gives the well-known identification of $H^*(G, R)$ with the cohomology of right invariant forms (form of Maurer-Cartan). On the other hand we see that each $c \in C^p(\mathfrak{G})$ corresponds to a unique $c' \in C_I^p(\mathfrak{G}, \mathfrak{T})$ such that $c(X_1, \cdots, X_p) = c'(X_1, \cdots, X_p)(e)$; namely $c'(X_1, \cdots, X_p)(g) = c(ad(g)X_1, \cdots, ad(g)X_p)$. This sets up a

linear isomorphism between $C^*(\mathfrak{G})$ and $C_I^*(\mathfrak{G}, \mathfrak{I})$ and referring to equations (5) and (6) of § 1 we see that d_T corresponds to $-d_I$ under this isomorphism. It follows that $H_I^*(\mathfrak{G}, \mathfrak{I}) \cong H^*(\mathfrak{G})$ and this gives the well-known result $H^*(G, R) \cong H^*(\mathfrak{G})$. In [2], where the cohomology theory of Lie algebras was first made explicit, there is a detailed account of theorems of this general nature.

11. Infinitesimal deformation of differentiable *G*-spaces. Let $\mathfrak{M}$ be a differentiable G-space, $\Phi\colon G \times \mathfrak{M} \to \mathfrak{M}$ the action of G on $\mathfrak{M}$ and $X \to X^*$ the infinitesimal geenrator of Φ. Suppose that for each $t \in [0,1] = I$ there is given an action Φ_t of G on $\mathfrak{M}$ such that $\Phi_0 = \Phi$ and such that $(g, p, t) \to \Phi_t(g, p)$ is a differentiable map of $G \times \mathfrak{M} \times I$ into $\mathfrak{M}$. Such a family Φ_t will be called a *deformation* of Φ and we write $X \to X_t^*$ for the infinitesimal generator of Φ_t. It is easily seen that $(X, p, t) \to (X_t^*)_p$ is a differentiable map of $\mathfrak{G} \times \mathfrak{M} \times I$ into the tangent bundle of $\mathfrak{M}$ and it follows that for each $X \in \mathfrak{G}$ there is a differentiable vector field DX on $\mathfrak{M}$ such that $(DX)_p = \dfrac{d}{dt}\Big|_{t=0} (X_t^*)_p$. Clearly D is a linear map of $\mathfrak{G}$ into the Lie algebra $\mathfrak{V}$ of differentiable vector fields on $\mathfrak{M}$ and since $[X, Y]_t^* = [X_t^*, Y_t^*]$ it follows that $D([X, Y]) = [DX, Y^*] + [X^*, DY]$. We call D the infinitesimal deformation of Φ associated with Φ_t and in general a linear map $D'\colon \mathfrak{G} \to \mathfrak{V}$ satisfying the above identity is called an infinitesimal deformation of Φ. Now if Φ_t is a deformation of $\mathfrak{M}$ (i. e. $(p, t) \to \Phi_t(p)$ is a differentiable map of $\mathfrak{M} \times I \to \mathfrak{M}$ and for each $t \in I$ ϕ_t is a diffeomorphism of $\mathfrak{M}$) the vector field Z defined by $Z_p =$ tangent to $t \to \Phi_t(p)$ at $t = 0$ is called the infinitesimal deformation of $\mathfrak{M}$ associated with ϕ_t. From ϕ_t we can construct a deformation Φ_t of Φ by $\Phi_t(g, p) = \phi_t(\Phi(g, \phi_t^{-1}(p)))$. A deformation of Φ that can be defined in this way is called *trivial*. It is easily seen that the associated infinitesimal deformation D of Φ is given by $DX = [Z, X^*] = ad(Z)X^*$ where Z as above is the infinitesimal deformation of $\mathfrak{M}$ associated with ϕ_t. In general if $Z' \in \mathfrak{V}D'\colon X \to ad(Z')X^*$ is an infinitesimal deformation of Φ and we call such infinitesimal deformations of Φ *trivial* (at least if $\mathfrak{M}$ is compact Z' is the infinitesimal deformation of $\mathfrak{M}$ associated with some deformation ϕ_t of $\mathfrak{M}$ so in this case D' is the infinitesimal deformation of Φ associated with a trivial deformation of Φ). Now consider $\mathfrak{V}$ as a tensor module associated to the G-space $\mathfrak{M}$ and recall that the Lie derivative of $Y \in \mathfrak{V}$ with respect to $Z \in \mathfrak{V}$ is just $[Z, Y]$ (see, for example [5] Lemma c). It follows that $\mathfrak{V}$ is a derived $\mathfrak{G}$ module under the operations $XY = [X^*, Y]$. Now an element of $C^1(\mathfrak{G}, \mathfrak{V})$ is just a linear map $c\colon \mathfrak{G} \to \mathfrak{V}$ and $c \in Z'(\mathfrak{G}, \mathfrak{V})$

if and only if $0 \equiv dc(X,Y) = Xc(Y) - Yc(X) - c([X,Y])$ i.e. if and only if $c([X,Y]) = [X^*, c(Y)] + [c(X), Y^*]$. Thus $Z'(\mathfrak{G}, \mathfrak{V})$ is just the space of infinitesimal deformations of Φ. On the other hand $c \in D'(\mathfrak{G}, \mathfrak{V})$ if and only if for some $Z \in \mathfrak{V} c(X) \equiv dZ(X) = X(Z) = [-Z, X^*]$ i.e. if and only if c is a trivial infinitesimal deformation of Φ. Thus $H^1(\mathfrak{G}, \mathfrak{V}) = 0$ means every infinitesimal deformation of Φ is trivial. Since $\mathfrak{V}$ is a tensor module it follows from the Theorem of § 9 that

11.1. THEOREM. *If G is a compact semi-simple Lie group then every infinitesimal deformation of the action of G on a differentiable G-space is trivial.*

In [7] the authors prove a global form of this theorem; namely that if G is any compact Lie group (not necessarily semi-simple) then every deformation of the action of G on a compact differentiable G-space is trivial. It does not seem that either of these theorems implies the other in any obvious way.

Now let $\mathfrak{L}$ be a finite dimensional subalgebra of the Lie algebra $\mathfrak{V}$ of all differentiable vector fields on a compact differentiable manifold $\mathfrak{M}$. If $\mathfrak{V}$ is a compact semi-simple Lie algebra (i.e. the Killing form is negative definite) then the simply connected Lie group G with Lie algebra $\mathfrak{G}$ isomorphic to $\mathfrak{L}$ is compact and semi-simple. By Corollary 2 of Theorem XVIII of [6] an isomorphism of $\mathfrak{G}$ onto $\mathfrak{L}$ is the infinitesimal generator of an action of G on $\mathfrak{M}$. Applying 11.1

11.2. THEOREM. *If $\mathfrak{L}$ is a compact semi-simple sub-algebra of the Lie algebra $\mathfrak{V}$ of differentiable vector fields on a compact differentiable manifold, then every derivation of $\mathfrak{L}$ into $\mathfrak{V}$ is the restriction to $\mathfrak{L}$ of an inner derivation of $\mathfrak{V}$.*

12. Lifting of group actions. In this section we assume that our compact Lie group G is connected and denote by $\tilde{G}$ its simply connected covering group. We shall identify the Lie algebras of G and $\tilde{G}$ under the isomorphism given by the differential of the covering homomorphism. We note that a differentiable G-space $\mathfrak{M}$ is in a natural way a differentiable $\tilde{G}$-space and that the homomorphisms of $\mathfrak{G}$ into the Lie algebra $\mathfrak{V}$ of differentiable vector fields on $\mathfrak{M}$ which are the infinitesimal generators of the actions of G and $\tilde{G}$ on $\mathfrak{M}$ are the same.

In general a homomorphism $X \to X^*$ of $\mathfrak{G}$ into $\mathfrak{V}$ is not the infinitesimal generator of an action of G or even $\tilde{G}$ on $\mathfrak{M}$ if $\mathfrak{M}$ is not compact. However it is shown in [6, Theorem III, p. 95] that if each of the vector fields X^* generates a global one-parameter group of diffeomorphism of $\mathfrak{M}$ then $X \to X^*$

is the infinitesimal generator of a unique action of $\tilde{G}$ on $\mathfrak{M}$. Now if X^* does not generate a global one parameter group of diffeomorphism of $\mathfrak{M}$ then (for example, see [6, p. 84 and p. 73]) there is an integral curve σ of X^* defined on an intevral $[0,a)$ or $(a,0]$ such that $\lim_{t\to a}\sigma(t)=\infty$ (i.e. for each compact subset K of $\mathfrak{M}$ $\sigma(t)\notin K$ for t sufficiently close to a). Suppose now Y is a vector field on a differentiable manifold $\mathfrak{N}$, X a vector field on $\mathfrak{M}$ and $f\colon \mathfrak{N}\to\mathfrak{M}$ is a differentiable map such that $df(Y_p)\equiv X_{f(p)}$ (under these circumstances we say, following Chevalley [1, p. 84] that Y and X are f-related). Then if σ is an integral curve of Y $f\circ\sigma$ is an integral curve of X and it follows from the above remark that if X generates a global one parameter group of diffeomorphisms of $\mathfrak{M}$ and if f is proper then Y generates a global one parameter group of diffeomorphisms of $\mathfrak{N}$.

Now let $X\to X^*$ be the infinitesimal generator of the action of G on a differentiable G-space $\mathfrak{M}$ and let π be the projection of a differentiable fiber bundle B over $\mathfrak{M}$, having compact fiber. Then π is proper and it follows that there is a one-to-one correspondence between actions of $\tilde{G}$ on B for which π is equivariant and homomorphisms $\tau\colon X\to X^\tau$ of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on $\mathfrak{M}$ such that X^τ and X^* are π-related for all $X\in\mathfrak{G}$. We now specialize further and assume that B is a principle-bundle with structural group a compact connected Lie group H and we write $(h,b)\to hb$ for the action of H on B (this conflicts with the more customary usage in which the structural group acts on the right, but it is only necessary to define hb to be bh^{-1}).

For a diffeomorphism of B to be a bundle map, i.e. equivariant with respect to the action of H, means just that it commutes with each operation of H; hence if $Z\to Z^*$ is the homomorphism of the Lie algebra $\mathfrak{H}$ of H into the differentiable vector fields on B which generates the action of H, then a one parameter group of diffeomorphism of B consists of bundle maps if and only if its infinitesimal generator Y satisfies $[Y,Z^*]=0$ for all $Z\in\mathfrak{H}$. Thus

12.1. THEOREM. *Let $X\to X^*$ be the infinitesimal generator of the action of G on a differentiable G-space $\mathfrak{M}$. Let H be a compact connected Lie group with Lie algsbra $\mathfrak{H}$, B a differentiable principle H-bundle over $\mathfrak{M}$ with projection π and $Z\to Z^*$ the infinitesimal generator of the action of H on B. Then there is a one-to-one correspondence between actions of $\tilde{G}$ on B equivariant with respect to π such that each operation of $\tilde{G}$ on B is a bundle equivalence, and homomorphisms $\tau\colon X\to X^\tau$ of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on B such that*

(1) *X^τ and X^* are π-related for all $X\in\mathfrak{G}$*

(2) *$[X^\tau,Z^*]=0$ for all $X\in\mathfrak{G}$, $Z\in\mathfrak{H}$.*

A homomorphism $\tau: X \to X^\tau$ satisfying (1) and (2) will be called a lifting of $\mathfrak{G}$ to B. A linear map $\tau: X \to X^\tau$ of $\mathfrak{G}$ into the Lie algebra of differentiable vector fields on B which satisfies these conditions will be called a *pseudo-lifting* of $\mathfrak{G}$ to B. To construct a pseudo-lifting of $\mathfrak{G}$ to B it is only necessary to choose an H-invariant Riemannian-metric for B and let $X_b{}^\tau$ be the unique vector at b orthogonal to the fibre which projects onto $X^*{}_{\pi(b)}$. It is easily checked that X^τ is differentiable and by construction it is π-related to X^*. That X^τ is H-invariant and hence commutes with Z^* for all $Z \in \mathfrak{H}$ follows from the invariance of the metric. Liftings of $\mathfrak{G}$ to B on the other hand need not always exist and one of the principle results of this section is that they in fact do exist (and are essentially unique) if H is a torus and G is semi-simple. In general we have the relation $Z^*{}_{hb} = dh(ad(h^{-1})Z)^*{}_b$ for $Z \in \mathfrak{H}$, $h \in H$. We now make a final simplifying assumption, namely that H is a torus so that it follows from the above relation that Z^* is an H-invariant vector field on B for all $Z \in \mathfrak{H}$. A vector field Y on B is called *vertical* if $\delta\pi(Y) = 0$ (i.e. Y is π-related to zero). Clearly for each $p \in \mathfrak{M}$ $Z \to Z^*|\pi^{-1}(p)$ is an isomorphism of $\mathfrak{H}$ with the space of vertical H-invariant vector fields on $\pi^{-1}(p)$. It follows that every vector field Y on B which is vertical and H-invariant is of the form $b \to (f(\pi(b)))^*{}_b$ for a uniquely determined function $f: \mathfrak{M} \to \mathfrak{H}$, moreover Y is differentiable if and only if f is differentiable (the latter meaning of course that $l \circ f$ is differentiable for every linear functional l on $\mathfrak{H}$). It follows that we may identify the space M of differentiable maps of $\mathfrak{M}$ into $\mathfrak{H}$ with the space of all vertical vector fields on B which are H-invariant (or, equivalently, which commute with Z^* for all $Z \in \mathfrak{H}$). We note that since $\mathfrak{H}$ is abelian any two elements of M, considered as vertical vector fields, commute. If Y is a vector field on $\mathfrak{M}$ and $m \in M$ then Ym is a well defined element of M; namely its value $(Ym)(p)$ at $p \in \mathfrak{M}$ is the unique element of $\mathfrak{H}$ such that $l((Ym)(p)) = Yp(l \circ m)$ for each linear functional l on $\mathfrak{H}$. Moreover by considering local product representations of B it is easily seen that if Y' is a vector field on B π-related to Y then $Ym = [Y', m]$. Clearly M becomes a $\mathfrak{G}$-module if we define $Xm = X^*{}_m$ for $X \in \mathfrak{G}$, $m \in M$. Moreover from the previous remark $Xm = [X^\tau, m]$ if $\tau: X \to X^\tau$ is any pseudo-lifting of $\mathfrak{G}$ to B. If $\mathfrak{F}$ is the tensor module of differentiable real valued functions on $\mathfrak{M}$ and $X_1, \cdots, X_n$ is a basis for $\mathfrak{H}$ then $(f_1, \cdots, f_n) \to f_1X_1 + \cdots + f_nX_n$ (where the latter means the elment of M whose value at p is $\sum f_i(p)X_i$) is an isomorphism (as $\mathfrak{G}$-modules) of the n-fold direct sum of $\mathfrak{F}$ with itself and M. It follows from the theorem of § 9 that $H^*(\mathfrak{G}, M) \cong H_I(\mathfrak{G}, M)$ and that if G is semi-simple then $H^1(\mathfrak{G}, M) = H^2(\mathfrak{G}, M) = 0$. We collect these results as

12.2. THEOREM. *With the assumptions of* 12.1 *and the additional assumption that H is a torus let M be the linear space of differentiable maps of $\mathfrak{M}$ into $\mathfrak{H}$. If we identify $m \in M$ with the vector field on B whose value at b is $(m(\pi(b))^*)_b$ then this gives a linear isomorphism of M with the linear space of differentialble vertical vector fields on B which commute with Z^* for all $Z \in \mathfrak{H}$. Considered as vertical vector fields on B any two elements of M commute. Moreover M is a $\mathfrak{G}$-module satisfying $H^*(\mathfrak{G}, M) = H^*_I(\mathfrak{G}, M)$ (and hence $H^1(\mathfrak{G}, M) = H^2(\mathfrak{G}, M) = 0$ if G is semi-simple) the module structure being characterized by the relation $Xm = [X^\tau, m]$* if $\tau\colon X \to X^\tau$ *is any pseudo-lifting of $\mathfrak{G}$ to B.*

Continuing now with the same assumptions, with each pseudo-lifting $\tau\colon X \to X^\tau$ of $\mathfrak{G}$ to B we associate a map c_τ of $\mathfrak{G} \times \mathfrak{G}$ into vector fields on B by $c_\tau(X, Y) = [X^\tau, Y^\tau] - [X, Y]^\tau$. Clearly c_τ is a measure of how much τ fails to be a lifting, i.e. $c_\tau \equiv 0$ if and only if τ is a lifting. Now $[X, Y]^\tau$ is π-related to $[X, Y]^*$ and by [1, p. 85] $[X^\tau, Y^\tau]$ is π-related to $[X^*, Y^*] = [X, Y]^*$. It follows that $c_\tau(X, Y)$ is π-related to 0, i.e. is vertical. Since $[X, Y]^\tau$ and $[X^\tau, Y^\tau]$ commute with Z^* for all $Z \in \mathfrak{H}$ ($[X, Y]^\tau$ by definition of a pseudo-lifting, $[X^\tau, Y^\tau]$ by the same plus the Jacobi identity) so does $c_\tau(X, Y)$, hence we can identify $c_\tau(X, Y)$ with an element of M. Moreover c_τ is clearly bilinear and skew-symmetric and hence an element of $C^2(\mathfrak{G}, M)$ If X_1, X_2, X_3 belong to $\mathfrak{G}$ then

$$\begin{aligned}[X_1^\tau, [X_2^\tau, X_3^\tau]] &= [X_1^\tau, [X_2, X_3]^\tau + c_\tau(X_2, X_3)] \\ &= [X_1, [X_2, X_3]]^\tau + c_\tau(X_1, [X_2, X_3]) + X_1 c_\tau(X_2, X_3)\end{aligned}$$

Now $[X_1^\tau, [X_2^\tau, X_3^\tau]]$ satisfies the Jacobi identity, i.e. its sum with its two cyclic permutations is zero. Since τ is linear so does $[X_1, [X_2, X_3]]^\tau$. Writing out the cyclic permutations of the above equation and summing we get an equation which gives $dc_\tau(X_1, X_2, X_3) = 0$, hence $c_\tau \in Z^2(\mathfrak{G}, M)$. We call c_τ the *error cocycle* of the pseudo-lifting τ.

Now let $\gamma \in C^1(\mathfrak{G}, M)$, i.e. γ is a linear map of $\mathfrak{G}$ into M. Then it is clear that $\sigma\colon X \to X^\sigma = X^\tau + \gamma(X)$ is another pseudo-lifting of $\mathfrak{G}$ to B and that conversely every pseudo lifting of $\mathfrak{G}$ to B is of this form for a unique $\gamma \in C^1(\mathfrak{G}, M)$. We call γ the difference cochain of σ and τ. Recalling from 12.2 that $[\gamma(X), \gamma(Y)] = 0$ for $X, Y \in \mathfrak{G}$

$$\begin{aligned}c_\sigma(X, Y) &= [X^\tau + \gamma(X), Y^\tau + \gamma(Y)] - [X, Y]^\tau - \gamma([X, Y]) \\ &= [X^\tau, Y^\tau] - [X, Y]^\tau - \gamma([X, Y] + X\gamma(Y) - Y\gamma(X) \\ &= c_\tau(X, Y) + d\gamma(X, Y)\end{aligned}$$

or in words, the difference of the error cocycles c_σ and c_τ is just the coboundary of the difference cochain $\gamma = \sigma - \tau$. Thus the set of error cocycles associated with pseudo-liftings of $\mathfrak{G}$ to B is an entire cohomology class $\omega \in H^2(\mathfrak{G}, M)$ which we call the *obstruction to a lifting of $\mathfrak{G}$ to B*. By what we have seen above a lifting of $\mathfrak{G}$ to B exists if and only if ω contains the zero element, thus we have the only apparently tautological statement

12.3. THEOREM. *If B is a principle torus bundle over a differentiable G-space $\mathfrak{M}$, then a lifting of $\mathfrak{G}$ to B exists if and only if the obstruction to a lifting of $\mathfrak{G}$ to B vanishes.*

Since $\omega \in H^2(\mathfrak{G}, M)$ which by 12.2 is zero if G is semi-simple.

12.4. COROLLARY. *If G is a compact semi-simple Lie group and B is a principle torus bundle over a differentiable G-space $\mathfrak{M}$ then there is a lifting of $\mathfrak{G}$ to B.*

We now consider the uniqueness problem for liftings of $\mathfrak{G}$ to B. Returning to our general situation let σ and τ be two liftings of $\mathfrak{G}$ to B. Their difference cochain γ is clearly a one-cocycle. Conversely $\gamma \in Z^1(\mathfrak{G}, M)$ and τ is a lifting of $\mathfrak{G}$ to B then so is $\sigma = \tau + \gamma$. Assume now that the obstruction cocycle ω is zero so that a lifting τ of $\mathfrak{G}$ to B exists and assume also that $H^1(\mathfrak{G}, M) = 0$ so that by the above remark every lifting σ of $\mathfrak{G}$ to B is of the form $\tau + dm$ for some $m \in M = C^0(\mathfrak{G}, M)$. More explicitly $X^\sigma = X^\tau + Xm = X^\tau + [X^\tau, m]$.

The most general bundle equivalence of B is of the form $b \to f(\pi(b))b$ when f is a differentiable map of $\mathfrak{M}$ into H. Since $\mathrm{Exp}\colon \mathfrak{H} \to H$ is the universal covering map of H it follows that every such f can be written in the form $\mathrm{Exp} \circ m$ for some $m \in M$. Moreover by taking a local product representation of B it is easily seen that if h is the bundle equivalence given by $b \to (\mathrm{Exp}(m(\pi(b)))b$ then $dh(X^\tau{}_{h^{-1}(b)}) = (X^\tau + Xm)_b$ i.e. $X^\sigma = dh \circ X^\tau \circ h^{-1}$. It follows that if $(g, b) \to gb$ is the action of $\tilde{G}$ on B generated by X^τ then the action generated by X^σ is $(gb) \to hgh^{-1}(\mathrm{b})$. Since $H^1(\mathfrak{G}, M) = 0$ if G is semi-simple combining these remarks with 12.1 and 12.4 we have

12.5. THEOREM. *If G is a semi-simple compact Lie group, $\mathfrak{M}$ a differentiable G-space, and B a torus bundle over $\mathfrak{M}$, then there is a differentiable action $(g, b) \to gb$ of $\tilde{G}$ on B which is equivariant with respect to the projection of B on $\mathfrak{M}$ and is such that each operation of $\tilde{G}$ on B is a bundle map. Moreover this action is essentially unique in the sense that every other such*

action of $\tilde{G}$ on B is of the form $(g,b) \to hgh^{-1}b$ where h is a differentiable bundle equivalence of B.

The latter theorem can be significantly generalized as follows

12.6. THEOREM. *Let G be a simply connected compact Lie group, H a solvable, connected Lie group, $\mathfrak{M}$ a differentiable G-space and B a differentiable principle H-bundle over $\mathfrak{M}$. Then there is a differentiable action of G on B such that the projection of B on $\mathfrak{M}$ is equivariant and each operation of G on B is a bundle map. Moreover this action is essentially unique in the sense that any other action of G on B with these properties is related by conjugation with a bundle equivalence as in 12.5.*

Proof. By induction on $\dim H$. If $\dim H = 1$ then either H is a circle group and the theorem is a consequence of 12.5 or else H is isomorphic to the additive group of real numbers. In the latter case H is solid so that [8, p. 55] B is a product bundle so that existence of the required type of action of G on B is obvious. Uniqueness can be proved just as in 12.5. Now suppose $\dim H > 1$ and that the theorem holds for all H of smaller dimension. Then H has a closed normal subgroup N such that both N and H/N are conected solvable Lie groups of dimension smaller than that of H. Now B is a principle N-bundle over the orbit space B/N and B/N is a principle H/N bundle over $\mathfrak{M}$ and the composition of the projections $B \to B/N \to \mathfrak{M}$ is just the bundle projection $B \to \mathfrak{M}$. By the induction hypothesis we can lift the action of G—first to B/N and then to B. Moreover any action of the apropriate sort on B induces one on B/N. Since the liftings from $\mathfrak{M}$ to B/N and from B/N to B are essentially unique by the induction hypothesis, the same follows easily for the lifting from $\mathfrak{M}$ to B.

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AN INVARIANT CRITERION OF HYPOELLIPTICITY.*

By François Treves.†

Introduction. A differential operator $\boldsymbol{P}$ on a C^∞ manifold Ω is said to be ***hypoelliptic*** in Ω if, whatever be the distribution u on Ω, u has to be a C^∞ function in any open set in which $\boldsymbol{P}u$ is a C^∞ function.

Several sufficient conditions of hypoellipticity have been given, from classical ones: the elliptic operators are hypoelliptic and so are the parabolic ones, to more recent (and weaker!) like the formal hypoellipticity, introduced by Lars Hörmander and Bernard Malgrange. Beside ellipticity, none of those sufficient conditions is invariant: they are properties of the expression of the operator in a system of coordinates which do not subsist when one changes coordinates.

Since the definition of hypoellipticity is itself obviously invariant, it was natural to look for invariant conditions. In the present paper, we define a class of differential operators, which we call *regular,* by invariant a priori estimates, and we prove that a regular operator is hypoelliptic.

This result can be considered only as a first step on a difficult road. For what one should wish, is to have properties of the expression of the differential operator in *any* system of local coordinates which insure hypoellipticity; and that we do not provide. However, in the last section, we prove rapidly that any formally hypoelliptic operator is regular.

The main step in our proof consists in deriving estimates about the commutator $[\boldsymbol{P}, \rho_\epsilon{}^*]$ of the differential operator $\boldsymbol{P}$ with Friedrichs mollifiers from "similar" estimates about the commutator $[\boldsymbol{P}, \boldsymbol{Q}]$ of $\boldsymbol{P}$ with arbitrary differential operators $\boldsymbol{Q}$ (§ 6). This could be of interest in questions which do not bear any relation to hypoellipticity.

Let us also mention that the reading of the article does not require any knowledge about distributions but the most superficial.

1. Differential operators and their derivatives. Let Ω be a C^∞ manifold of dimension $n \geqq 1$. As usual $C^\infty(\Omega)$ will denote the vector space of C^∞ complex valued functions on Ω. A differential operator $\boldsymbol{P}$ on Ω is a linear map of $C^\infty(\Omega)$ into itself which satisfies the following conditions:

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$(DO_I)_I$ For every compact subset K of Ω there is an integer $M_K \geqq 0$ such that if all the derivatives of $\phi \in C^\infty(\Omega)$ of order $\leqq M_K$ converge to zero uniformly on K, $\boldsymbol{P}\phi \to 0$ uniformly on K.

(DO_{II}) The support of $\boldsymbol{P}\phi$ is contained in the support of ϕ.[1]

We denote by ω the variable point in Ω. If T is a letter, ϕ any element of $C^\infty(\Omega)$, it follows from (DO_I) that, on every compact subset of Ω,

$$e^{-T\phi(\omega)}\boldsymbol{P}(e^{T\phi(\omega)})$$

is a polynomial in T of degree $m(\phi,\omega)$ depending on ϕ and on the point ω; $m(\omega) = \sup m(\phi,\omega)$ is *the order of* $\boldsymbol{P}$ *at* ω. For any subset A of Ω, $\sup_{\omega\in A} m(\omega)$ is *the order of* $\boldsymbol{P}$ *in* A; observe that it could be infinite (it is certainly finite when A is compact). In this article we shall *not* make the assumption that $m(\omega)$ is independent of ω.

Let Ω' be an open subset of Ω covered by local coordinats $(x^1,\cdots,x^n)$. In Ω' we may look at the partial differentiations $\dfrac{\partial}{\partial x^1},\cdots,\dfrac{\partial}{\partial x^n}$ and their products which we shall write

$$D_\alpha = \left(\frac{1}{2i\pi}\,\frac{\partial}{\partial x^{\alpha_1}}\right)\cdots\left(\frac{1}{2i\pi}\,\frac{\partial}{\partial x^{\alpha_r}}\right);$$

the α_j are integers and $1 \leqq \alpha_j \leqq n$; we set $\alpha = (\alpha_1,\cdots,\alpha_r)$ and r, which is called the *length* of α, will be denoted by $|\alpha|$. If $\phi \in C^\infty(\Omega')$, then for $x \in \Omega'$

$$\boldsymbol{P}\phi(x) = \sum A_\alpha(x) D_\alpha\phi(x);$$

here x can be regarded as the vector $(x^1,\cdots,x^n)$; the coefficients $A_\alpha(x)$ are C^∞ functions of x; for every x the summation is performed over a finite set of multi-indices α, actually over those α satisfying $|\alpha| \leqq m(x)$. The "differential operator" $\sum A_\alpha(x)D_\alpha$ is the *expression* of $\boldsymbol{P}$ in the coordinates $(x^1,\cdots,x^n)$; we shall usually denote it by $P(x,D)$.

To the expression $P(x,D)$ we may associate the polynomial (with coefficients in $C^\infty(\Omega')$)

$$P(x,\xi) = \sum A_\alpha(x)\xi_\alpha, \qquad \xi_\alpha = \xi_{\alpha_1}\cdots\xi_{\alpha_r};$$

and $\deg P(x,\xi) = m(x)$. If $\beta = (\beta_1,\cdots,\beta_s)$, $1 \leqq \beta_j \leqq n$, we set:

$$P^{(\beta)}(x,\xi) = \left(\frac{1}{2i\pi}\,\frac{\partial}{\partial\xi_{\beta_1}}\right)\cdots\left(\frac{1}{2i\pi}\,\frac{\partial}{\partial\xi_{\beta_s}}\right)P(x,\xi);$$

$$P_{(\beta)}(x,\xi) = \left(\frac{\partial}{\partial x^{\beta_1}}\right)\cdots\left(\frac{\partial}{\partial x^{\beta_s}}\right)P(x,\xi).$$

[1] Mr. J. Peetre has proved that (DO_{II}) implies (DO_I). See J. Peetre, "Une caractérisation abstraite des opérateurs différentiels," *Math. Scand.*, vol. 7 (1959), p. 211.

If $\boldsymbol{Q}$ is another differential operator in Ω', these notations allow us to express the commutator $[\boldsymbol{P},\boldsymbol{Q}]$ in the coordinates $x^1,\cdots,x^n$:

$$(1.0)\qquad [\boldsymbol{P},\boldsymbol{Q}]=\sum_{|\beta|+|\gamma|>0}\frac{(-1)^{|\gamma|}}{|\beta|!\,|\gamma|!}Q_{(\beta)}{}^{(\gamma)}(x,D)\,P_{(\gamma)}{}^{(\beta)}(x,D).$$

The proof of (1.0) is of course elementary. Let us derive from it two important particular cases. First we take $\boldsymbol{Q}$ of order zero; $\boldsymbol{Q}$ is then simply the multiplication by a C^∞ function ψ (on Ω'). Then (see Hörmander [1]):

$$(1.1)\qquad [\boldsymbol{P},\boldsymbol{Q}]=\sum_{|\beta|>0}\frac{1}{|\beta|!}\;\frac{\partial\psi}{\partial x^\beta}P^{(\beta)}(x,D).$$

We use here, for simplicity, the notation

$$\frac{\partial}{\partial x^\beta}=\frac{\partial}{\partial x^{\beta_1}}\cdots\frac{\partial}{\partial x^{\beta_s}};$$

we have $\dfrac{\partial}{\partial x^\beta}=(2i\pi)^{|\beta|}D_\beta$.

Second, let us assume that the expression of $\boldsymbol{Q}$ in the coordinates x^j is a polynomial $Q(D)$ in $\dfrac{\partial}{\partial x^1},\cdots,\dfrac{\partial}{\partial x^n}$ with *constant* coefficients. Then:

$$(1.2)\qquad [\boldsymbol{P},\boldsymbol{Q}]=\sum_{|\beta|>0}\frac{(-1)^{|\beta|}}{|\beta|!}\;Q^{(\beta)}(D)P_{(\beta)}(x,D).$$

We want to derive from (1.1) and (1.2) two other formulas which will be useful later.

For that, we need to put an order relation (actually, a pre-order relation) on the multi-indices β. Let $\alpha=(\alpha_1,\cdots,\alpha_r)$, $\beta=(\beta_1,\cdots,\beta_s)$, $1\leqq\alpha_j\leqq n$, $1\leqq\beta_k\leqq n$ be two such multi-indices. We say that $\beta\leqq\alpha$ if, for every $p=1,\cdots,n$, the number of β_k equal to p is $\leqq$ number of α_j equal to p. If $\alpha\leqq\beta$ and $\beta\leqq\alpha$, we write $\alpha\sim\beta$: α and β are then identical modulo a permutation. If $\beta\leqq\alpha$ without $\alpha\leqq\beta$, we write $\beta<\alpha$. With those notations:

$$\boldsymbol{P}(x^\alpha\phi)=\sum_{\beta\leqq\alpha}\frac{1}{|\beta|!}\;\frac{\partial}{\partial x^\beta}(x^\alpha)P^{(\beta)}(x,D)\phi,\ \phi\in C^\infty(\Omega'),\ x^\alpha=x^{\alpha_1}\cdots x^{\alpha_r}.$$

Also:

$$\sum_{\beta\sim\alpha}\frac{1}{|\beta|!}\;\frac{\partial}{\partial x^\beta}\;(x^\alpha)P^{(\beta)}(x,D)\phi=P^{(\alpha)}(x,D)\phi.$$

Therefore:

$$(1.3)\qquad P^{(\alpha)}(x,D)=[\boldsymbol{P},x^\alpha]-\sum_{0<\beta<\alpha}\frac{1}{|\beta|!}\;\frac{\partial}{\partial x^\beta}(x^\alpha)P^{(\beta)}(x,D).$$

Similarly, by applying (1.2) this time:

$$(1.4)\quad (-1)^{|\alpha|} P_{(\alpha)}(x,D) = [\mathbf{P}, D_\alpha] - \sum_{0<\beta<\alpha} \frac{(-1)^{|\beta|}}{|\beta|!} (D_\alpha)^{(\beta)} P_{(\beta)}(x,D).$$

2. The norms $\| \ \|_s$. We consider now the n-dimensional real space R^n with Cartesian coordinates $x^1,\cdots,x^n$; we set $|x| = (|x^1|^2 + \cdots + |x^n|^2)^{\frac{1}{2}}$; $dx = dx^1 \cdots dx^n$. Let us denote by $C_0^\infty(R^n)$ the subset of $C^\infty(R^n)$ consisting of functions with compact support. We define the Fourier transform of $\phi \in C_0^\infty(R^n)$ as

$$\hat{\phi}(\xi) = \int_{R^n} \phi(x) e^{-2i\pi\langle x,\xi\rangle} dx,$$

where ξ belongs to the dual R_n of R^n and $\langle x,\xi\rangle = x^1\xi_1 + \cdots + x^n\xi_n$. The reciprocity formula holds:

$$\phi(x) = \int_{R_n} \hat{\phi}(\xi) e^{2i\pi\langle x,\xi\rangle} d\xi.$$

We set $|\xi| = (|\xi_1|^2 + \cdots + |\xi_n|^2)^{\frac{1}{2}}$. Let s be any real number. We set:

$$\|\phi\|_s = [\int_{R^n} (1 + |\xi|^2)^s |\hat{\phi}(\xi)|^2 d\xi]^{\frac{1}{2}}, \quad \phi \in C_0^\infty(R^n).$$

The norms $\| \ \|_s$ are invariant in the following sense: Let η be an automorphism of the C^∞ structure of R^n; $\phi \to \phi\circ\eta$ is a one-to-one mapping of $C_0^\infty(R^n)$ onto itself. For every compact subset K of R^n there is a constant $C_K < +\infty$ such that, if $\phi \in C_0^\infty(R^n)$ and if support of $\phi \subset K$,

$$\|\phi\circ\eta\|_s \leqq C_K \|\phi\|_s$$

(see Peetre [1], p. 18).

Let us go back to the manifold Ω. For any open subset Ω' of Ω, we denote by $C_0^\infty(\Omega')$ the C^∞ functions in Ω whose supports are compact and contained in Ω'. Assume that Ω' is covered by local coordinates $x^1,\cdots,x^n$ and let J be the diffeomorphism they define from Ω' onto an open subset O' of R^n; then $\phi \to \phi\circ J^{-1}$ is a one-to-one linear map of $C_0^\infty(\Omega')$ onto $C_0^\infty(O')$. We may define the norm $\|\phi\|_s$ of $\phi \in C_0^\infty(\Omega')$ by equating it to $\|\phi\circ J^{-1}\|_s$. Of course $\|\phi\|_s$ depends on the coordinates x^j; but if we change coordinates and denote by $|||\phi|||_s$ the new corresponding norm, to every compact subset K of Ω' corresponds a constant $C_K < +\infty$ such that

$$C_K^{-1} \|\phi\|_s \leqq |||\phi|||_s \leqq C_K \|\phi\|_s$$

provided that the support of ϕ lies in K. This is a consequence of the invariance of the norms $\| \ \|_s$ in R^n we have indicated above.

Let now K be an arbitrary compact set in Ω. We may construct a partition of unity $\alpha_1, \cdots, \alpha_m$ in $C_0^\infty(\Omega)$ over some neighborhood of K such that the support of each α_j lies in some local map Ω'_j. In Ω'_j we may define the norms $\| \ \|_s$ and then, for $\phi \in C^\infty(\Omega)$ with support in K, we may set;

$$\| \phi \|_s = \{ \| \alpha_1 \phi \|_s^2 + \cdots + \| \alpha_m \phi \|_s^2 \}^{\frac{1}{2}}.$$

If one changes the partition of unity or the local coordinates, one gets an equivalent norm. Anyway, let C_K^∞ be the set of $\phi \in C_0^\infty(\Omega)$ having their support in K. The completion of C_K^∞ for the norm $\| \phi \|_s$ is denoted by $H^s{}_K$; $H^s{}_K$ is a Hilbert space (but the Hilbert structure depends on the partition of unity $\{\alpha_j\}$ and on the local coordinates [2]).

Let us assume that we have used the same partition of unity and the same local coordinates in defining $H^s{}_K$ whatever be s real. Then we can state the following properties:

1) The identity mapping of C_K^∞ can be extended, when $s \geqq t$, as a one-to-one continuous linear map $H^s{}_K \to H^t{}_K$, in fact of norm $\leqq 1$, which we shall call the natural embedding of $H^s{}_K$ into $H^t{}_K$.

Lemma 1. *If $s > t$ and $s \geqq 0$, the norm of the natural embedding $H^s{}_K \to H^t{}_K$ tends to zero when the diameter of K tends to zero.*

For the proof, see e. g. Malgrange [1] (Lemma I. 2. 3).

2) Let Q be a differential operator defined and of order $\mu \geqq 0$ on some neighborhood of K. Then, for $\phi \in C_K^\infty$,

$$\| Q\phi \|_s \leqq A_K(s, Q) \| \phi \|_{s+\mu}$$

where $A_K(s, Q) < +\infty$ depends on s, Q, K but not on ϕ. Hence $\phi \to Q\phi$, which maps C_K^∞ into itself, expands as a bounded linear map $H^{s+\mu}{}_K \to H^s{}_K$. The case $\mu = 0$ is important: Q is then the multiplication by a C^∞ function ψ on some neighborhood of K and, for $\phi \in C_K^\infty$,

$$\| \psi\phi \|_s \leqq A_K(s, \psi) \| \phi \|_s.$$

All this is valid whatever be s real.

When $\Omega = R^n$, we may define, for s real, the completion H^s of $C_0^\infty = C_0^\infty(R^n)$ for the norm $\phi \to \| \phi \|_s$. We shall make use of different properties of the spaces H^s:

i) for $t \leqq s$, $H^s \subset H^t$; H^s is dense in H^t; the norm of the embedding $H^s \to H^t$ is $\leqq 1$;

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[2] However, as a topological vector space, $H^s{}_K$ is invariant.

ii) for $\phi, \psi \in C_0^\infty$, the bilinear functional

$$(\phi,\psi) \to \langle \phi,\psi \rangle = \int_{R^n} \phi(x)\psi(x)\,dx$$

can be extended to $H^s \times H^{-s}$ and turns those spaces into the duals of each other.

Now for a few words about distributions in Ω. The elements of $H^s{}_K$ can be canonically regarded as such distributions, whose supports, morevoer, lie in K. If u is a distribution with compact support K_u, we shall use the notation $u \in H^s$ to indicate that $u \in H^s{}_K$ whatever be K compact, $K \supset K_u$. Similarly, we shall use expressions like: a set of distributions u bounded, compact, etc. in H^s; a net of distributions converging in H^s; etc. This applies only, when Ω is an arbitrary manifold, in our set up, to sets or nets of distributions having all their support in some fixed compact subset K of Ω and means that the properties in question really occur in $H^s{}_K$.

Given any distribution u on Ω and any $\phi \in C_0^\infty(\Omega)$, there is a real number s such that $\phi u \in H^s$. In general, s depends on ϕ. If s can be taken independently of ϕ, we say that $u \in H^s{}_{\mathrm{loc}}(\Omega)$. In other words, given any distribution v in Ω and any open set $\Omega' \subset \Omega$, $\overline{\Omega'}$ compact, there is a real q such that the restriction of v to Ω' belongs to $H^q{}_{\mathrm{loc}}(\Omega')$. If $u \in H^s{}_{\mathrm{loc}}(\Omega)$ whatever be s real, u is a C^∞ function on Ω.

Let us finally add that all the differential operators will be assumed, in the sequel, to act in the sense of distributions.

3. Some lemmas. Let now ρ be a C_0^∞ function on R^n, $\int_{R^n} \rho(x)\,dx = 1$. For $t > 0$, we set: $\rho_t(x) = \frac{1}{t^n}\rho(\frac{x}{t})$. The Fourier transform of ρ_t is

$$(\rho_t)^\wedge(y) = \int_{R^n} e^{-2i\pi\langle x,y\rangle}\rho(\frac{x}{t})\frac{dx}{t^n} = \hat{\rho}(ty).$$

LEMMA 2. *For every real number μ' there is a number $\mu \geqq 0$ such that, for all multi-indices $\beta = (\beta_1, \cdots, \beta_r)$, $1 \leqq \beta_j \leqq n$ for $j = 1, \cdots, r$, such that $|\beta| \geqq \mu$, we have, for all numbers $0 < t < 1$,*

$$\| x^\beta \rho_t \|_{\mu'} \leqq C(|\beta|)$$

where $C(|\beta|) < +\infty$ depends only on ρ and $|\beta|$.

Let $\mu'' = \sup(\mu', 0)$.

$$\begin{aligned}\| x^\beta \rho_t \|^2_{\mu'} &\leqq \int (1 + |y|^2)^{\mu''} |(D_\beta)_y \hat{\rho}(ty)|^2\,dy \\ &= t^{2|\beta|} \int (1 + |y|^2)^{\mu''} |(D_\beta \hat{\rho})(ty)|^2\,dy \\ &\leqq t^{2|\beta|-2\mu''} \int (1 + |ty|^2)^{\mu''} |(D_\beta \hat{\rho})(ty)|^2\,dy \\ &= t^{2|\beta|-2\mu''-n} \int (1 + |\xi|^2)^{\mu''} |D_\beta \hat{\rho}(\xi)|^2\,d\xi\end{aligned}$$

and therefore any $\mu \geqq \mu'' + \frac{n}{2}$ has the desired property.

For $\psi \in C_0^\infty$, we set:

$$\| \psi \|_{\sigma,\infty} = \sup_{\xi \in R_n} (1 + |\xi|^2)^{\sigma/2} | \hat{\psi}(\xi) |, \ \sigma \text{ real.}$$

If $\phi \in C_0^\infty$ also, then:

$$\| \phi * \psi \|_\sigma \leqq \| \phi \|_{\sigma'} \| \psi \|_{\sigma'', \infty} \tag{3.1}$$

whatever be the real numbers σ', σ'' such that $\sigma' + \sigma'' = \sigma$.

LEMMA 3. *Let β be a multi-index: $\beta = (\beta_1, \cdots, \beta_r)$, $1 \leqq \beta_j \leqq n$ for $j = 1, \cdots, r$. Whatever be $0 < t < 1$,*

$$\| x^\beta \rho_t \|_{|\beta|,\infty} \leqq C(|\beta|)$$

where $C(|\beta|) < +\infty$ depends only on ρ and $|\beta|$.

We have:

$$\begin{aligned}
&\sup_{y \in R_n, 0<t<1} (1 + |y|^2)^{|\beta|/2} | (D_\beta)_y \hat{\rho}_t(y) | \\
&\quad = \sup_{y \in R_n, 0<t<1} t^{|\beta|} (1 + |y|^2)^{|\beta|/2} | (D_\beta \hat{\rho})(ty) | \\
&\quad \leqq \sup_{y \in R_n, 0<t<1} (1 + |ty|^2)^{|\beta|/2} | (D_\beta \hat{\rho})(ty) | \\
&\quad = \sup_{\xi \in R_n} (1 + |\xi|^2)^{|\beta|/2} | (D_\beta \hat{\rho})(\xi) | < +\infty
\end{aligned}$$

(the Fourier transform of a C_0^∞ function decreases, as well as each one of its derivatives, when $|\xi| \to +\infty$, faster than any power of $\frac{1}{|\xi|}$).

COROLLARY. *Let σ be any real number. Whatever be $0 < t < 1$ and $\phi \in C_0^\infty$,*

$$\| (x^\beta \rho_t) * \phi \|_\sigma \leqq C(|\beta|) \| \phi \|_{\sigma - |\beta|}.$$

Immediate consequence of Formula (3.1) and Lemma 3.

LEMMA 4. *Let O be an open set in R^n, $R(x;y)$ a C^∞ function in $O \times O$ whose derivatives, up to the order $k-1$ included ($k \geqq 1$), vanish on the diagonal $x = y$. Then one may write*

$$R(x;y) = \sum_{|\alpha|=k} (x-y)^\alpha R_\alpha(x;y)$$

where the $R_\alpha(x,y)$ are C^∞ functions in $O \times O$.

We recall that $(x-y)^\alpha = (x^{\alpha_1} - y^{\alpha_1}) \cdots (x^{\alpha_r} - y^{\alpha_r})$ where $1 \leqq \alpha_j \leqq n$

for $j=1,\cdots,r$; $\alpha=(\alpha_1,\cdots,\alpha_r)$. It is enough to prove the Lemma for $k=1$; by iteration, one gets it then for arbitrary k. Let us therefore assume that $R(x;y)=0$ if $x=y$. For $j=1,\cdots,n$, let us set:

$$R_j(x;y)=\frac{1}{x^j-y^j}\{R(x^1,\cdots,x^j,y^{j+1},\cdots,y^n;y^1,\cdots,y^n)\\ -R(x^1,\cdots,x^{j-1},y^j,\cdots,y^n;y^1,\cdots,y^n)\}.$$

Since $R(y^1,\cdots,y^n;y^1,\cdots,y^n)=0$, we have

$$R(x;y)=\sum_{j=1}^{n}(x^j-y^j)R_j(x;y).$$

We have to show that the $R_j(x;y)$ are C^∞ in $O\times O$. But it is enough to show it when $n=1$, in which case the result is trivial.

Let $K(x,u)$ be a C_0^∞ function on $R_x{}^n\times R_u{}^n$. We set:

$$[\phi\underset{K}{\otimes}\psi](x)=\int_{R^n}\phi(x-u)K(x,u)\psi(u)\,du,$$

where $\phi,\psi\in C^\infty=C^\infty(R^n)$.

LEMMA 5. *Let σ, τ be two real numbers. For every $\phi,\psi\in C_0{}^\infty$,*

$$\|\phi\underset{K}{\otimes}\psi\|_{\sigma+\tau}\leq C(K;\sigma,\tau)\|\phi\|_\sigma\|\psi\|_\tau;$$

$C(K;\sigma,\tau)$ is a constant $<+\infty$ depending on K, σ, τ but not on ϕ, ψ.

Of course, $C_0{}^\infty=C_0{}^\infty(R^n)$. We set $s=\sigma+\tau$.

We may write $K(x,y)=\int e^{2i\pi(\langle x,y\rangle+\langle u,v\rangle)}\hat{K}(y,v)\,dydv$, where $\hat{K}(y,v)$ is a function belonging to $\mathcal{S}_{y,v}$.* In particular, whatever be the integer $k\geqq 0$ the function $(1+|y|+|v|)^k\hat{K}(y,v)$ is square integrable in $R_y{}^n\times R_v{}^n$. Let us set

$$H(x,y,v)=\int e^{2i\pi(\langle x,y\rangle+\langle u,v\rangle)}\phi(x-u)\psi(u)\,du.$$

We have:

$$[\phi\underset{K}{\otimes}\psi](x)=\int\hat{K}(y,v)H(x,y,v)\,dydv. \tag{3.2}$$

It is obvious that $H(x,y,v)$ is C^∞ in x, y, v, and, regarded as a function of x, has its support in a fixed compact set independent of y, v. On the other hand, $H(x,y,v)$ is bounded in R^{3n}. From this follows that the Fourier transform of the right hand side of (3.2) equals

$$\int\hat{K}(y,v)\hat{H}(\xi,y,v)\,dydv,$$

* $\mathcal{S}_w$ is the space of C^∞ functions of $w\in R^M$ decreasing at infinity, as well as each one of their derivatives, faster than any power of $1/|w|$.

where $\hat{H}(\xi, y, v)$ is the Fourier transform of $H(x, y, v)$ with respect to x. We have then, in view of Schwarz inequality,

$$(3.3)\quad (1+|\xi|^2)^s|(\phi \underset{K}{\otimes} \psi)^\wedge(\xi)|^2 \leqq \left(\int (1+|y|+|v|)^k|\hat{K}(y,v)|^2\, dydv\right) \times \left(\int (1+|y|+|v|)^{-k}(1+|\xi|^2)^s|\hat{H}(\xi,y,v)|^2\, dydv\right),$$

where k is any integer $\geqq 0$. We shall show now that under the hypotheses of Lemma 5, there is an integer $N \geqq 0$ such that, for all $y, v \in R_n$,

$$\int (1+|\xi|^2)^s|\hat{H}(\xi,y,v)|^2 d\xi \leqq C(1+|y|+|v|)^N \|\phi\|^2{}_\sigma \|\psi\|^2{}_\tau.$$

Choosing then $k > N + 2n$ and integrating with respect to ξ both sides of (3.3), we obtain the lemma.

We have $\hat{H}(\xi, y, v) = \hat{\phi}(\xi - y)\hat{\psi}(\xi - y - v)$. Let us now use the fact that for every real number r there is a finite constant C_r such that for all $\xi, \eta \in R_n$,

$$(1+|\xi|^2)^r \leqq C_r(1+|\eta|^2)^{|r|}(1+|\xi-\eta|^2)^r.$$

We get:

$$(1+|\xi|^2)^s|\hat{H}(\xi,y,v)|^2 \leqq C(1+|y|^2)^{|\sigma|}(1+|y+v|^2)^{|\tau|} \times (1+|\xi-y|^2)^\sigma|\hat{\phi}(\xi-y)|^2(1+|\xi-y-v|^2)^\tau|\hat{\psi}(\xi-y-v)|^2.$$

By integrating the two sides of this inequality, we get

$$\int (1+|\xi|^2)^s|\hat{H}(\xi,y,v)|^2 d\xi \leqq C_1(1+|y|+|v|)^{2(|\sigma|+|\tau|)} \times \sup_{\xi \in R_n}(1+|\xi|^2)^\tau|\hat{\psi}(\xi)|^2 \cdot \|\phi\|^2{}_\sigma.$$

At that point it should be noted that since $K(x, u)$ has its support in a fixed compact subset of $R_x{}^n \times R_u{}^n$, we may as well assume that the support of ψ lies in a fixed compact set $\mathcal{K}$. Let $\alpha \in C_0{}^\infty(R^n)$, $\alpha \equiv 1$ on $\mathcal{K}$. We have $\hat{\psi} = \hat{\psi} * \hat{\alpha}$ and therefore

$$(1+|\xi|^2)^{\tau/2}|\hat{\psi}(\xi)| \leqq C\int (1+|\eta|^2)^{|\tau|/2}|\hat{\alpha}(\eta)| \times (1+|\xi-\eta|^2)^{\tau/2}|\hat{\psi}(\xi-\eta)|\, d\eta \leqq C_2\|\psi\|_\tau,$$

by applying Schwarz inequality. This concludes the proof of Lemma 5.

4. Statement of the main result. Let Ω be an n-dimensional C^∞ manifold.

In order to abbreviate the forthcoming statements, we shall *constantly* make use of the following convention.

Let $\boldsymbol{R}$, $\boldsymbol{S}$ be two differential operators defined near a point ω of Ω (i.e. on some open neighborhood of ω). We shall make statements of the following kind:

The inequality $\| \boldsymbol{R}\phi \|_r \leqq C \| \boldsymbol{S}\phi \|_s$ *is valid at* ω.

The meaning of it is the following:

Let Ω' be an open neighborhood of ω on which both $\boldsymbol{R}$ and $\boldsymbol{S}$ are defined.

i) There is an open neighborhood $U \subset \Omega'$, depending on $\boldsymbol{R}$, $\boldsymbol{S}$, r, s and the local coordinates near ω used to define the norms $\| \ \|_\sigma$,

ii) there is a finite constant C, depending on $\boldsymbol{R}$, $\boldsymbol{S}$, r, s and the local coordinates, such that, for every $\phi \in C_0^\infty(\Omega)$ with support in U,

$$\| \boldsymbol{R}\phi \|_r \leqq C \| \boldsymbol{S}\phi \|_s.$$

Of course C is independent of ϕ; we shall handle C as a kind of general symbol.

In order to state the main result of this article, we shall introduce the following definition:

Let $\omega \in \Omega$, $\boldsymbol{P}$ be a differential operator defined on Ω. We shall always denote by m the order of $\boldsymbol{P}$ on some open subset of Ω containing ω.

Let σ, a, b be real numbers. Let us consider the following properties which may be valid at the point ω of Ω:

(A_σ)
$$\| \phi \|_{\sigma+a} \leqq C \| \boldsymbol{P}\phi \|_\sigma.$$

(B_σ) *Whatever be the differential operator* $\boldsymbol{L}$ *defined and of order* $\mu \leqq 1$ *near* ω,

$$\| [\boldsymbol{P}, \boldsymbol{L}]\phi \|_{\sigma+b} \leqq C \| \boldsymbol{P}\phi \|_{\sigma+\mu}.$$

DEFINITION 1. *We say that* $\boldsymbol{P}$ *is regular at* ω *if there are numbers* σ_0 *real*, a *real*, $b > 0$ *such that* (A_σ), (B_σ) *hold at* ω *for every* $\sigma \geqq \sigma_0$.

We say that $\boldsymbol{P}$ is regular in some open subset Ω_0 of Ω if $\boldsymbol{P}$ is regular at every point ω of Ω_0 (σ_0, a, b will then generally depend on the point ω).

Remark 4.1. If (A_σ) is valid at ω, a must necessarily be $\leqq m$; $a = m$ if and only if $\boldsymbol{P}$ is elliptic in some neighborhood of ω.

If (B_σ) holds at ω, b must be $\leqq 1$ and $b = 1$ if and only if $\boldsymbol{P}$ is elliptic near ω.

Our main theorem will then be:

THEOREM 1. *If* $\boldsymbol{P}$ *is regular in* Ω, $\boldsymbol{P}$ *is hypoelliptic in* Ω.

We shall prove a slightly more precise statement:

THEOREM 1′. *Let* $\boldsymbol{P}$ *be regular in* Ω; *let* Ω_1 *be an open subset of* Ω *with compact closure. There is a real number* a_1 *such that, whatever be* s *real and the distribution* u *on* Ω,

$$\boldsymbol{P}u \in H^s{}_{\mathrm{loc}}(\Omega_1) \text{ implies } u \in H^{s+a_1}{}_{\mathrm{loc}}(\Omega_1).$$

5. First consequences of (A_σ), (B_σ). Let us introduce the following property, which may be valid at $\omega \in \Omega$:

$(B_s)_M{}^{\#}$ *Whatever be the $\lambda \leqq M$ differential operators $L_1, \cdots, L_\lambda$ of respective orders $d_1 \leqq 1, \cdots, d_\lambda \leqq 1$ near ω,*

$$\| [\cdots [P, L_1], \cdots L_\lambda]\phi \|_{s-d+b} \leqq C \| P\phi \|_s$$

with $d = d_1 + \cdots + d_\lambda$.

LEMMA 6. *Assume that (B_s) holds at ω, with $b \geqq 0$, for $s = \sigma, \sigma + 1, \cdots, \sigma + M$ $(M \geqq 1)$. Then $(B_{\sigma+M})_M{}^{\#}$ holds at ω.*

Let us set $R_\lambda = [\cdots [P, L_1], \cdots L_\lambda]$. We are going to show that

$$\| R_\lambda \phi \|_{s-d+b} \leqq C \| P\phi \|_s \text{ at } \omega \tag{5.1}$$

for $s = \sigma + \lambda, \sigma + \lambda + 1, \cdots, \sigma + M$. For $\lambda = M$, this will be exactly what we want.

We reason by induction on λ. When $\lambda = 1$, (5.1) is a trivial consequence of the validity of (B_s) for $s = \sigma, \sigma + 1, \cdots, \sigma + M$. We assume $\lambda \geqq 2$ and that (5.1) is valid up to $\lambda - 1$. We have $R_\lambda = [R_{\lambda-1}, L_\lambda]$ and therefore, at ω, setting $d' = d_1 + \cdots + d_{\lambda-1}$:

$$\begin{aligned} \| R_\lambda \phi \|_{s-bd+} &\leqq \| R_{\lambda-1} L_\lambda \phi \|_{s-d_\lambda-d'+b} + \| L_\lambda R_{\lambda-1} \phi \|_{s-d'+b-d_\lambda} \\ &\leqq \| R_{\lambda-1} L_\lambda \phi \|_{s-d_\lambda-d'+b} + C \| R_{\lambda-1} \phi \|_{s-d'+b}. \end{aligned}$$

Assume that s belongs to the set of numbers $\{\sigma + \lambda, \cdots, \sigma + M\}$. Then s and $s - d_\lambda$ (since $d_\lambda \leqq 1$) both belong to the set

$$\{\sigma + \lambda - 1, \cdots, \sigma + M\};$$

hence at ω:

$$\| R_{\lambda-1} L_\lambda \phi \|_{s-d_\lambda-d'+b} \leqq C \| P L_\lambda \phi \|_{s-d_\lambda},$$

$$\| R_{\lambda-1} \phi \|_{s-d'+b} \leqq C \| P\phi \|_s.$$

Since $b \geqq 0$,

$$\begin{aligned} \| P L_\lambda \phi \|_{s-d_\lambda} &\leqq \| L_\lambda P\phi \|_{s-d_\lambda} + \| [P, L_\lambda]\phi \|_{s-d_\lambda} \\ &\leqq C \| P\phi \|_s + \| [P, L_\lambda]\phi \|_{s-d_\lambda+b}. \end{aligned}$$

But certainly $s \in \{\sigma, \cdots, \sigma + M\}$; hence at ω

$$\| [P, L_\lambda]\phi \|_{s-d_\lambda+b} \leqq C \| P\phi \|_s.$$

This concludes the proof.

LEMMA 7. *Assume that (A_s) holds at ω. Let M_0 be the smallest integer $\geqq 2m - a + b$. If $(B_s)_{M_0}{}^{\#}$ holds at ω, $(B_s)_M{}^{\#}$ holds at ω whatever be the integer M.*

We keep the same notations as in the proof of Lemma 6. Let $x^1, \cdots, x^n$ be local coordinates near ω. Let $P(x,D)$, $R_\lambda(x,D)$ be the respective expressions of $\boldsymbol{P}$ and $\boldsymbol{R}_\lambda$ in those coordinates. We may write

$$R_\lambda(x,D) = \sum R_{\lambda,\alpha}{}^\beta(x,D)\, P_{(\beta)}{}^{(\alpha)}(x,D),$$

where the $R_{\lambda,\alpha}{}^\beta(x,D)$ are differential operators depending on $\boldsymbol{L}_1, \cdots, \boldsymbol{L}_\lambda$. Observe that in the summation, because of the definition of $\boldsymbol{R}_\lambda$, we will always have $|\alpha| \geqq \lambda - d$, $|\alpha| + |\beta| \geqq \lambda$.

Assume $\lambda \geqq M \geqq 2m - a + b$. Then we must have $d \geqq m - a + b$ or else $\boldsymbol{R}_\lambda \equiv 0$ near ω. For if $d < m - a + b$, $|\alpha| \geqq \lambda - d > m$ and $P_{(\beta)}{}^{(\alpha)} \equiv 0$ near ω.

If $d \geqq m - a + b$, $\| \phi \|_{s-d+m+b} \leqq \| \phi \|_{s+a}$.

Observe on the other hand that the order of $\boldsymbol{R}_\lambda$ near ω is $\leqq m$ whatever be λ. Hence at ω

$$\| \boldsymbol{R}_\lambda \phi \|_{s-d+b} \leqq C \| \phi \|_{s-d+m+b}$$

and by combining those two inequalities, we get at ω

$$\| \boldsymbol{R}_\lambda \phi \|_{s-d+b} \leqq C \| \phi \|_{s+a}.$$

Then the Lemma is simply a consequence of (A_s).

If $(B_s)_M{}^\#$ holds at ω whatever be M, we say that $(B_s)^\#$ holds at ω.

LEMMA 8. *Property $(B_s)^\#$ is valid at ω if and only if the following property is valid at ω:*
(C_s) *Whatever be the differential operator $\boldsymbol{Q}$ defined and of order μ near ω,*

$$\| [\boldsymbol{P}, \boldsymbol{Q}]\phi \|_{s-\mu+b} \leqq C \| \boldsymbol{P}\phi \|_s.$$

1) $\underline{(B_s)^\# \Longrightarrow (C_s)}$

Let us use again the local coordinates x^j; let $Q(x,D)$ be the expression of $\boldsymbol{Q}$ in those coordinates. According to Formula (1.0) we have at ω:

$$\| [\boldsymbol{P}, \boldsymbol{Q}]\phi \|_{s-\mu+b} \leqq C \sum_{|\beta|+|\gamma|>0} \| P_{(\gamma)}{}^{(\beta)}(x,D)\phi \|_{s-|\gamma|+b},$$

since the order of $Q_{(\beta)}{}^{(\gamma)}(x,D)$ near ω is $\leqq \mu - |\gamma|$.

Observe that if $1 \leqq \lambda \leqq n$, $1 \leqq \lambda' \leqq n$,

$$[[\boldsymbol{P}, x^\lambda], \frac{\partial}{\partial x^{\lambda'}}] = [[\boldsymbol{P}, \frac{\partial}{\partial x^{\lambda'}}], x^\lambda]$$

and therefore $P_{(\gamma)}{}^{(\beta)}(x,D)$ is the expression in the coordinates x^j of

$$[\cdot\cdot\cdot[\cdot\cdot\cdot[[\boldsymbol{P},x^{\beta_1}],\cdot\cdot\cdot x^{\beta_r}],\frac{\partial}{\partial x^{\gamma_1}}],\cdot\cdot\cdot\frac{\partial}{\partial x^{\gamma_s}}]$$

assuming that $\beta=(\beta_1,\cdot\cdot\cdot,\beta_r)$, $\gamma=(\gamma_1,\cdot\cdot\cdot,\gamma_s)$. Then, from $(B_s)^{\#}$, it follows immediately that, at ω

$$\| P_{(\gamma)}{}^{(\beta)}(x,D)\phi \|_{s-|\gamma|+b} \leqq C \| \boldsymbol{P}\phi \|_s.$$

2) $\underline{(C_s) \Longrightarrow (B_s)^{\#}}$

It is easily seen that $[\cdot\cdot\cdot[\boldsymbol{P},\boldsymbol{L}_1],\cdot\cdot\cdot \boldsymbol{L}_\lambda]$ is a sum of terms of the form

$$\pm \boldsymbol{L}_{i_1}\cdot\cdot\cdot \boldsymbol{L}_{i_\alpha}[\boldsymbol{P},\boldsymbol{L}_{j_1}\cdot\cdot\cdot \boldsymbol{L}_{j_\beta}]$$

where $(i_1,\cdot\cdot\cdot,i_\alpha,j_1,\cdot\cdot\cdot,j_\beta)$ is simply a permutation of $(1,2,\cdot\cdot\cdot,\lambda)$. We have at ω:

$$\begin{aligned}&\| \boldsymbol{L}_{i_1}\cdot\cdot\cdot \boldsymbol{L}_{i_\alpha}[\boldsymbol{P},\boldsymbol{L}_{j_1}\cdot\cdot\cdot \boldsymbol{L}_{j_\beta}]\phi \|_{s-d+b}\\ &\qquad\leqq C \| \boldsymbol{P},\boldsymbol{L}_{j_1}\cdot\cdot\cdot \boldsymbol{L}_{j_\beta}]\phi \|_{s-d^0+b} \leqq C' \| \boldsymbol{P}\phi \|_s,\end{aligned}$$

where $d=d_1+\cdot\cdot\cdot+d_\lambda$, $d^0=d_{j_1}+\cdot\cdot\cdot+d_{j_\beta}$; $\boldsymbol{L}_{j_1}\cdot\cdot\cdot \boldsymbol{L}_{j_\beta}$ is precisely a differential operator of order d^0 near ω.

COROLLARY. *Let $x^1,\cdot\cdot\cdot,x^n$ be local coordinates near ω, $P(x,D)$ the expression of $\boldsymbol{P}$ in those coordinates.*

If (C_s) is true at ω, then whatever be the multi-indices α,β, $|\alpha|+|\beta| > 0$,

$$\| P_{(\beta)}{}^{(\alpha)}(x,D)\phi \|_{s-|\beta|+b} \leqq C \| \boldsymbol{P}\phi \|_s$$

holds at ω.

The important result in this section is the following:

LEMMA 9. *Assume that (A_σ), (B_σ) hold at ω, with $b\geqq 0$, for every $\sigma \geqq \sigma_0$. Let M_0 be the smallest integer $\geqq 2m-a+b$. Let $s_0=\sigma_0+M_0$.*

Then (C_s) holds at ω for every $s\geqq s_0$.

Lemma 6 tells us that $(B_{\sigma'})$, valid at ω for $\sigma\leqq\sigma'\leqq\sigma+M_0$, implies $(B_{\sigma+M_0})_{M_0}{}^{\#}$ at ω.

Lemma 7 tells us that $(B_{\sigma+M_0})_{M_0}{}^{\#} \Longrightarrow (B_{\sigma+M_0})^{\#}$ provided that $(A_{\sigma+M_0})$ is true at ω.

Lemma 8 state that $(B_{\sigma+M_0})^{\#}$ is equivalent with $(C_{\sigma+M_0})$.

Q. E. D.

Remark 5.1. From now on, instead of working on the basis that (A_σ) and (B_σ) are valid at ω for $\sigma \geqq \sigma_0$, we shall assume that (A_s) and (C_s) are valid at ω for $s \geqq s_0$.

We conclude this section with a trivial result that is going to be needed later on.

LEMMA 10. *Assume that (C_s) holds at ω. Let Ω' be some open set containing ω. There is an open neighborhood $\Omega_s'' \subset \Omega'$ of ω such that for every $\psi \in C^\infty(\Omega')$ and every $\phi \in C_0^\infty(\Omega''_s)$,*

$$\| [\boldsymbol{P}, \psi] \phi \|_{s+b} \leqq C \| \boldsymbol{P} \phi \|_s$$

This means that the inequality in the Lemma is valid for $\phi \in C_0^\infty(\Omega)$ having its support in a neighborhood of ω independent of ψ.

Proof. Let $x^1, \cdots, x^n$ be a system of coordinates covering an open neighborhood $\Omega_1' \subset \Omega'$ of ω; let $P(x, D)$ be the expression of $\boldsymbol{P}$ in those coordinates. If $\phi \in C_0^\infty(\Omega_1')$, we have, according to Formula (1.1):

$$\| [\boldsymbol{P}, \psi] \phi \|_{s+b} \leqq C \sum_{|\beta| > 0} \| P^{(\beta)}(x, D) \phi \|_{s+b}$$

(in fact, we have to assume something like $\overline{\Omega_1'}$ compact, $\overline{\Omega_1'} \subset \Omega'$). For every multi-index β, $|\beta| > 0$, we have according to the Corollary of Lemma 8,

$$\| P^{(\beta)}(x, D) \phi \|_{s+b} \leqq C \| \boldsymbol{P} \phi \|_s$$

for $\phi \in C_0^\infty(\Omega'_{(\beta)})$, where $\Omega'_{(\beta)}$ is a suitable neighborhood of ω. Observe that $P^{(\beta)}(x, D) \equiv 0$ near ω for $|\beta| > m$. We may take then

$$\Omega''_s = \Omega'_1 \cap \bigcap_{0 < |\beta| \leqq m} \Omega'_{(\beta)}.$$

6. Properties (A_s), (C_s) and Friedrichs mollifiers. We choose local coordinates $x^1, \cdots, x^n$ near ω; we shall always assume that $x^1(\omega) = \cdots = x^n(\omega) = 0$. The norms $\| \ \|_s$ are computed by means of those coordinates; $P(x, D)$ will be the expression of $\boldsymbol{P}$ in the x^j.

Let ρ be a C_0^∞ function on R^n, $\int_{R^n} \rho(x) dx = +1$. For $t > 0$, we set:

$$\rho_t(x) = t^{-n} \rho(x/t).$$

Let J be the homeomorphism $\omega' \to (x^1(\omega'), \cdots, x^n(\omega'))$ of a neighborhood Ω' of ω onto an open neighborhood O' of the origin $0 \in R^n$; $J(\omega) = 0$. The map $\phi \to \phi \circ J^{-1}$ is *one-to-one* from $C_0^\infty(\Omega')$ *onto* $C_0^\infty(O')$. Let $\Omega'' \subset \Omega'$ be

open, $\overline{\Omega''}$ compact, $\overline{\Omega''} \subset \Omega'$. We may find $t_0 > 0$ small enough so that $t \leqq t_0$ implies, for $\phi \in C_0^\infty(\Omega'')$, that the support of

$$[\rho_{t*}(\phi \circ J^{-1})](x) = \int_{R^n} \rho_t(x - x')\phi(J^{-1}x')\,dx'$$

lies in O'. Therefore the support of $[\rho_{t*}(\phi \circ J^{-1})] \circ J$ will lie in Ω'.

We still denote by ρ_{t*} the operator $C_0^\infty(\Omega'') \to C_0^\infty(\Omega')$

$$\phi \to [\rho_{t*}(\phi \circ J^{-1}| \circ J.$$

This operator ρ_{t*} can be extended as a linear operator from the space of distributions in Ω, with compact support contained in Ω'', into $C_0^\infty(\Omega')$. In particular (and that is what we are going to need later on), if $u \in H^s{}_K$, K compact, $K \subset \Omega''$, $\rho_{t*}u$ is a C_0^∞ function (with support in Ω'). Moreover, when $t > 0$ tends to 0, $\rho_{t*}u \to u$ in the sense of H^s.

From now on we assume that the number t_0 defined above equals 1. We may always be in that case by just replacing $\rho(x)$ by $t_0^{-n}\rho(x/t_0)$.

LEMMA 11. *Assume that* $(A\sigma)$, $(C\sigma)$ *hold at* ω. *Then*

$$\| [\boldsymbol{P}, \rho_{t*}]\phi \|_{\sigma+b} \leqq C \| \boldsymbol{P}\phi \|_\sigma$$

holds at ω *independently of* $0 < t < 1$.

By "independently of t" we mean the following: i) the constant $C < +\infty$ does not depend on t; ii) we may take an open neighborhood N of ω, independent of t, such that the inequality is true for every $\phi \in C_0^\infty(N)$. Here, of course, both C and N depend (in general) on the function ρ.

Proof. According to Lemma 4, we may write, for some integer $\mu \geqq 1$,

$$P(x, D) = \sum_{|\beta| < \mu} \frac{(x-u)^\beta}{|\beta|!} P_{(\beta)}(u, D_x) + \sum_{|\beta| = \mu} (x-u)^\beta R_\beta(x, u, D_x)$$

where the $R_\beta(x, u, D_x)$ are differential operators on O'. The coefficients of $R_\beta(x, u, D_x)$ are C^∞ functions of $(x, u) \in O' \times O'$; the order of $R_\beta(x, u, D_x)$ is $\leqq m$.

If $O'' =$ image of Ω'' under J and $\psi \in C_0^\infty(O'')$, we have:

$$P(x, D)(\rho_{t*}\psi) = P(x, D_x)\int \rho_t(x-u)\psi(u)\,du$$

$$= \sum_{|\beta| < \mu} \frac{1}{|\beta|!}(x^\beta\rho_t)_* P_{(\beta)}(x, D_x)\psi + \sum_{|\beta| = \mu} \int (x-u)^\beta \rho_t(x-u) R_\beta(x, u, D_u)\psi(u)\,du.$$

We have taken into account the fact that

$$Q(D_x)(\rho_{t*}\psi) = \int \rho_t(x-u)Q(D_u)\psi(u)\,du,$$

where $Q(D)$ stands for any polynomial (with constant coefficients!) in $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$.

According to the Corollary of Lemma 3,

$$\| (x^\beta \rho_t)_* P_{(\beta)}(x,D)\psi \|_{\sigma+b} \leqq C \| P_{(\beta)}(x,D)\psi \|_{\sigma+b-|\beta|}$$

independently of t. Let us apply then the corollary of Lemma 8. For $|\beta| > 0$, we have at ω:

$$\| P_{(\beta)}(x,D)\phi \|_{\sigma+b-|\beta|} \leqq C \| P\phi \|_\sigma$$

where $\phi = \psi \circ J$.[3] We are therefore left with the estimation of the norm $\| g \|_{\sigma+b}$ of the function

$$g(x) = \sum_{|\beta|=\mu} \int (x-u)^\beta \rho_t(x-u) R_\beta(x,u,D_u)\psi(u)\,du,$$

which is a sum of terms of the form

$$g^\alpha(x) = \int (x-u)^\beta \rho_t(x-u) R_\beta{}^\alpha(x,u)(D_\alpha\psi)(u)\,du.$$

There is a compact set $H \subset R^n$, such that $O'' + H \subset O'$ and that the supports, in $R_x{}^n \times R_u{}^n$, of the functions $\rho_t(x-u)(D_\alpha\psi)(u)$ (whatever be α and $0 < t < 1$) are contained in $(O'' + H) \times O''$. Hence we may assume that the functions $R_\beta{}^\alpha(x,u)$ have their supports contained in a fixed compact subset of $R_x{}^n \times R_u{}^n$. This allows to apply Lemma 5:

$$\| g^\alpha \|_{\sigma+b} \leqq C \| x^\beta \rho_t \|_{\mu'} \| D_\alpha\psi \|_{\sigma+b-\mu'}$$

where μ' is any real number (of course, C depends on μ'). On the other hand,

$$\| D_\alpha\psi \|_{\sigma+b-\mu'} \leqq C \| \psi \|_{\sigma+m+b-\mu'}.$$

We choose $\mu' = m - a + b$. According to Property (A_σ):

$$\| \phi \|_{\sigma+m+b-\mu'} = \| \phi \|_{\sigma+a} \leqq C \| \boldsymbol{P}\phi \|_\sigma \text{ at } \omega,$$

where $\phi = \psi \circ J$.

---

[3] $P_{(\beta)}(x,D)$ is a differential operator on the open subset O of R^n. Through the diffeomorphism J, it defines a differential operator on $\Omega' \subset \Omega$ whose expression, in the coordi- $x^1, \cdots, x^n$, is $P_{(\beta)}(x,D)$.

Ultimately, we must show that we can choose the integer μ in such a way that the numbers $\| x^\beta \rho_t \|_{\mu'}$, $|\beta| = \mu$, are bounded independently of t. This follows from Lemma 2.

Remark 6.1. No assumption whatsoever has been made upon the sign of b: we did not need $b > 0$ or even $b \geqq 0$; b could be any real number.

7. The last lemma. In this section and the next one, we assume that $\boldsymbol{P}$ has properties (A_s) and (C_s) at ω, with $b > 0$ and for every $s \geqq s_0$ (s_0 being some real number). We recall that those properties are:

(A_s) $\qquad \| \phi \|_{s+a} \leqq C \| \boldsymbol{P}\phi \|_s.$

(C_s) *Whatever be the differential operator $\boldsymbol{Q}$ of order μ near ω,*

$$\| [\boldsymbol{P}, \boldsymbol{Q}] \phi \|_{s-\mu+b} \leqq C \| \boldsymbol{P}\phi \|_s.$$

It is essential that a real and $b > 0$ should be independent of $s \geqq s_0$.

The three numbers s_0, a, b give, in some sense, a "measure" of the regularity of $\boldsymbol{P}$ near ω. They will serve to determine the number a_1 of Theorem 1′ and some other regularity properties of $\boldsymbol{P}$—as we are going to see. In view of those properties, let us say that a and b should be as large as possible while s_0 should be as *small* as possible. We recall however that a cannot exceed m and b cannot exceed 1 (Remark 4.1). We shall actually use those facts.

Anyway, in order to abbreviate the forthcoming statements, we shall say that, under our assumption, ***P*** *is of type* (s_0, a, b) *at* ω.

Let $x^1, \cdots, x^n$ be local coordinates near ω. We introduce the operator

$$U_p = \left(1 - \frac{\Delta}{4\pi^2}\right)^p, \quad p\text{: integer} \geqq 0.$$

$\Delta = \left(\dfrac{\partial}{\partial x^1}\right)^2 + \cdots + \left(\dfrac{\partial}{\partial x^n}\right)^2$; U_p is defined on some open set containing ω.

LEMMA 12. *If* ***P*** *is regular at* ω, *so is* $\boldsymbol{P}U_p$. *More precisely, if* ***P*** *is of type* (s_0, a, b), $\boldsymbol{P}U_p$ *is of type* $(s_0^+ - 2p + b, a + 2p, b)$ *at* ω, *with* $s_0^+ = \sup(s_0, 0)$.

Let $\sigma \geqq s_0^+$. We have at ω:

$$\begin{aligned} \| \boldsymbol{P}U_p\phi \|_{\sigma-2p+b} &\geqq \| U_p\boldsymbol{P}\phi \|_{\sigma-2p+b} - \| [\boldsymbol{P}, U_p]\phi \|_{\sigma-2p+b} \\ &\geqq \| \boldsymbol{P}\phi \|_{\sigma+b} - C \| \boldsymbol{P}\phi \|_\sigma. \end{aligned}$$

Since $\sigma \geqq 0$, $b > 0$, we may apply Lemma 1: if the support of ϕ lies in a suitable neighborhood of ω,

$$C \| \boldsymbol{P}\phi \|_{\sigma} \leqq \tfrac{1}{2} \| \boldsymbol{P}\phi \|_{\sigma+b}.$$

Therefore, at ω:

$$\| \boldsymbol{P}\phi \|_{\sigma+b} \leqq C \| \boldsymbol{P}U_p\phi \|_{\sigma-2p+b}. \tag{7.1}$$

Let us prove first (A_s) for $\boldsymbol{P}U_p$ and $s \geqq s_0{}^+ - 2p + b$ (moreover, with $a+2$ instead of a). We have at ω, for $\sigma \geqq s_0$:

$$\| \phi \|_{\sigma+b+a} \leqq C \| \boldsymbol{P}\phi \|_{\sigma+b} \text{ (since } b \geqq 0).$$

By applying (7.1) and setting $\sigma = s + 2p - b$, we get at ω:

$$\| \phi \|_{s+(a+2p)} \leqq C \| \boldsymbol{P}U_p\phi \|_s$$

which is what we wanted.

Let us prove now (C_s) for $\boldsymbol{P}U_p$ and $s \geqq s_0{}^+ - 2p + b$. Let $\boldsymbol{Q}$ be any differential operator of order μ near ω. We have

$$[\boldsymbol{P}U_p, \boldsymbol{Q}] = [\boldsymbol{P}, U_p\boldsymbol{Q}] + \boldsymbol{Q}[U_p, \boldsymbol{P}] + [U_p, \boldsymbol{Q}]\boldsymbol{P}.$$

Note that $[U_p, \boldsymbol{Q}]$ is a differential operator of order $\leqq 2p + \mu - 1$ near ω. Hence we have at ω:

$$\| [\boldsymbol{P}_p, \boldsymbol{Q}]\phi \|_{s-\mu+b} \leqq \| [\boldsymbol{P}, U_p\boldsymbol{Q}]\phi \|_{s-\mu+b} + C(\| [U_p, \boldsymbol{P}]\phi \|_{s+b} + \| \boldsymbol{P}\phi \|_{s+2p});$$

in order to obtain the last term of the left hand side, we have used the fact that $b \leqq 1$.

Let us take $s = \sigma - 2p + b$, $\sigma \geqq s_0{}^+$. The order of $U_p\boldsymbol{Q}$ near ω being $2p + \mu$, we have at ω, because of (C_σ) for $\boldsymbol{P}$:

$$\| [\boldsymbol{P}, U_p\boldsymbol{Q}]\phi \|_{\sigma+b-(2p+\mu)+b} \leqq C \| \boldsymbol{P}\phi \|_{\sigma+b}.$$

On the other hand, at ω:

$$\| [\boldsymbol{P}, U_p]\phi \|_{\sigma+b-2p+b} \leqq C \| \boldsymbol{P}\phi \|_{\sigma+b}.$$

We have therefore, at ω:

$$\| [\boldsymbol{P}U_p, \boldsymbol{Q}]\phi \|_{\sigma+b-2p-\mu+b} \leqq C \| \boldsymbol{P}\phi \|_{\sigma+b}.$$

Then (7.1) gives the result.

Q. E. D.

Let us now take mollifiers ρ_t as in § 6. We shall consider the following property of a distribution u in Ω:

$(R)_\sigma$ *There is an open neighborhood N_σ of ω such that the following is true, for $\phi \in C_0^\infty(N_\sigma)$:*

i) $\boldsymbol{P}(\phi u) \in H^\sigma$;

ii) *when* $t \to 0$, $\boldsymbol{P}[\rho_{t*}(\phi u)] \to \boldsymbol{P}(\phi u)$ *weakly in* H^σ.

LEMMA 13. *Let $\boldsymbol{P}$ be regular of type (s_0, a, b) at ω. If $u \in H^{s_0+m}{}_{\mathrm{loc}}(\Omega)$ and $\boldsymbol{P}u \in H^s{}_{\mathrm{loc}}(\Omega)$, then $(R)_s$ holds.*

Since $u \in H^{s_0+m}{}_{\mathrm{loc}}(\Omega)$, $(R_{s_0}$ is trivially true. Let us assume that $(R)_\sigma$ holds for some $\sigma \geqq s_0$. Of course we assume $s > \sigma$.

Let Ω''_σ be the open neighborhood of ω considered in Lemma 10 (where we take $\Omega' = \Omega$). Let us set

$$N'_\sigma = \Omega''_\sigma \cap N_\sigma.$$

Let U, V be two open neighborhoods of ω, with the following properties: $\bar{U}$, $\bar{V}$ are compact, $\bar{U} \subset V$, $\bar{V} \subset N'_\sigma$. If $\phi_1 \in C_0^\infty(V)$, we may find a number $t_0 > 0$ such that, for $t \leqq t_0$, $\rho_{t*}(\phi_1 u)$ has its support in N'_σ. Let $\phi \in C_0^\infty(U)$ and let us assume that $\phi_1 = 1$ on a neighborhood of $\bar{U}$. We have:

$$\boldsymbol{P}(\phi u) = \phi \boldsymbol{P} u + [\boldsymbol{P}, \phi](\phi_1 u).$$

We have then, for $t < t_0$,

$$\| [\boldsymbol{P}, \phi]\{\rho_{t*}(\phi_1 u)\} \|_{\sigma+b} \leqq C \| \boldsymbol{P}\{\rho_{t*}(\phi_1 u)\} \|_\sigma.$$

According to $(R)_\sigma$ the right hand side is uniformly bounded with respect to $t \to 0$. We use now the fact that closed bounded set are weakly compact in $H^{\sigma+b}$: the functions $[\boldsymbol{P}, \phi]\{\rho_{t*}(\phi_1 u)\}$ must weakly converge in $H^{\sigma+b}$ as $t \to 0$. Their limit can only be $[\boldsymbol{P}, \phi](\phi_1 u)$, which therefore belongs to $H^{\sigma+b}$.

Let c be a number $\geqq 0$, $c \leqq b$, $c \leqq s - \sigma$. Since $\sigma + c \leqq s$, $\phi \boldsymbol{P} u \in H^{\sigma+c}$; since $c \leqq b$, $[\boldsymbol{P}, \phi](\phi_1 u) \in H^{\sigma+c}$; hence $\boldsymbol{P}(\phi u) \in H^{\sigma+c}$.

This being done, we apply Lemma 11. There is an open neighborhood W of ω and a constant $C < +\infty$, both independent of $t \leqq t_0$ ($t_0 > 0$ sufficiently small), such that, for $\psi \in C_0^\infty$ with support in W,

$$\begin{aligned}
\| \boldsymbol{P}(\rho_{t*}\psi) \|_{\sigma+c} &\leqq \| [\boldsymbol{P}, \rho_{t*}]\psi \|_{\sigma+c} + \| \rho_{t*}\boldsymbol{P}\psi \|_{\sigma+c} \\
&\leqq \| [\boldsymbol{P}, \rho_{t*}]\psi \|_{\sigma+b} + \| \rho_{t*}\boldsymbol{P}\psi \|_{\sigma+c} \\
&\leqq C \| \boldsymbol{P}\psi \|_\sigma + \| \rho_{t*}\boldsymbol{P}\psi \|_{\sigma+c}.
\end{aligned}$$

Let U' be an open neighborhood of ω, such that U' is compact and $U' \subset W$. Let us set $\tilde{N}_{\sigma+c} = U \cap U'$. If $\phi \in C_0^\infty(\Omega)$ has its support in $\tilde{N}_{\sigma+c}$, $\psi = \rho_{\epsilon*}(\phi u)$ has its support in W as soon as $\epsilon > 0$ is small enough. On the other hand, $\boldsymbol{P}(\phi u) \in H^{\sigma+c}$.

When $\epsilon \to 0$,

i) $\rho_{t*}\boldsymbol{P}\psi \to \rho_{t*}\boldsymbol{P}(\phi u)$ in $H^{\sigma+c}$ trivially;

ii) $\mathbf{P}(\rho_{t*}\psi) \to \boldsymbol{P}(\rho_{t*}(\phi u))$ in $H^{\sigma+c}$ trivially;

iii) $\boldsymbol{P}\psi \to \boldsymbol{P}(\phi u)$ weakly in H^σ because of $(R)_\sigma$.

At the limit, we have:

$$\| \boldsymbol{P}(\rho_{t*}(\phi u)) \|_{\sigma+c} \leqq C \| \boldsymbol{P}(\phi u) \|_\sigma + \| \rho_{t*}\boldsymbol{P}(\phi u) \|_{\sigma+c}.$$

When $t \to 0$, $\rho_{t*}\boldsymbol{P}(\phi u) \to \boldsymbol{P}(\phi u)$ in $H^{\sigma+c}$ since $\boldsymbol{P}(\phi u) \in H^{\sigma+c}$. We derive from there that $\boldsymbol{P}(\rho_{t*}(\phi u))$ remains bounded in $H^{\sigma+c}$, therefore converges weakly in $H^{\sigma+c}$, necessarily to $\boldsymbol{P}(\phi u)$. We have thus proved that whatever be $s_0 \leqq \sigma \leqq s$, $(R)_\sigma \Longrightarrow (R)_{\sigma+c}$ for $c \geqq 0$, $c \leqq b$, $c \leqq s - \sigma$. Take an integer ν large enough so that $c = (s - s_0)/\nu \leqq b$. We have proved that for every $\mu = 0, \cdots, \nu - 1$, $(R)_{s_0+\mu c} \Longrightarrow (R)_{s+(\nu+1)c}$. For $\mu = \nu - 1$, we get $(R)_s$.

LEMMA 14. *Same hypotheses as in Lemma* 13 *for* $\boldsymbol{P}$, u, $\boldsymbol{P}u$. *Then* ω *has an open neighborhood* U *such that* $\phi \in C_0^\infty(U)$ *implies* $\phi u \in H^{s+a}$.

Let us assume $s \geqq s_0$, for if s were $< s_0$, $s + a$ would be $< s_0 + m$ (since always $a \leqq m$) and since $u \in H^{s_0+m}{}_{\mathrm{loc}}(\Omega)$, there would be nothing to prove.

We may first choose U in such a way that for t sufficiently small,

$$\| \rho_{t*}(\phi u) \|_{s+a} \leqq C \| \boldsymbol{P}(\rho_{t*}(\phi u)) \|_s.$$

If we take $U \subset N_s$ (neighborhood of ω, used in Property $(R)_s$), we see that the right hand side is uniformly bounded with respect to t. We derive from there that $\rho_{t*}(\phi u)$ converges weakly in H^{s+a} as $t \to 0$, necessarily to ϕu.

8. End of the proof of Theorem 1′. We make the assumption that $\boldsymbol{P}u \in H^s{}_{\mathrm{loc}}(\Omega)$ (u: distribution on Ω). It is enough to prove the following: let ω be an arbitrary point of Ω and assume that $\boldsymbol{P}$ is of type $(s_0(\omega), a(\omega), b(\omega))$

at ω (according to Remark 5.1). Then there is an open neighborhood $U(\omega)$ of ω such that

$$\phi \in C_0^\infty(U(\omega)) \Longrightarrow \phi u \in H^{s+a(\omega)}.$$

For let then Ω_1 be an open subset of Ω, with compact closure. We may cover Ω_1 with a finite number of such open sets $U(\omega_1), \cdots, U(\omega_r)$ and take

$$a_1 = \inf(a(\omega_1), \cdots, a(\omega_r)).$$

In other words, we must extend the conclusion of Lemma 14 to an arbitrary distribution u in Ω. But if Ω_2 is open, $\bar{\Omega}_2$ compact, $\bar{\Omega}_1 \subset \Omega_2$, there is a real number q such that the restriction of u to Ω_2 belongs to $H_{\mathrm{loc}}{}^q(\Omega_2)$. We may therefore assume that $u \in H^q{}_{\mathrm{loc}}(\Omega)$.

Let ω be any point of Ω; let $x^1, \cdots, x^n$ be local coordinates near ω and let us introduce the operator

$$U_p = (1 - \frac{\Delta}{4\pi^2})^p, \quad p\text{: integer} \geqq 0.$$

We assume that $\boldsymbol{P}$ is of order (s_0, a, b) at ω. Let us choose p large enough so that $q + 2p \geqq s_0^+ + m + b$, with $s_0^+ = \sup(s_0, 0)$. There is a distribution v on some neighbrohood Ω' of ω such that $u = U_p v$ in Ω'. And $v \in H^{q+2p}{}_{\mathrm{loc}}(\Omega') \subset H^{s_0^+ + m + b}{}_{\mathrm{loc}}(\Omega')$.

We have $\boldsymbol{P}U_p v \in H^s{}_{\mathrm{loc}}(\Omega')$. We know that $\boldsymbol{P}U_p$ is of type $(s_0^+ - 2p + b, a + 2p, b)$ (Lemma 12). Let us set $\tau_0 = s_0^+ - 2p + b$. We see that $v \in H^{\tau_0 + (m+2p)}{}_{\mathrm{loc}}(\Omega')$ and $m + 2p$ is precisely the order of $\boldsymbol{P}U_p$ near ω. We therefore apply Lemma 14, with $\boldsymbol{P}U_p$ instead of $\boldsymbol{P}$, Ω' instead of Ω, τ_0 instead of s_0, etc. There is an open neighborhood Ω^* of ω such that $\phi \in C_0^\infty(\Omega^*) \Longrightarrow \phi v \in H^{s+(a+2p)}$.

Let $\phi_1 \in C_0^\infty(\Omega^*)$ be identically one on some open neighborhood $\Omega^{**} \subset \Omega^*$ of ω; $\phi_1 v \in H^{s+(a+2p)}$. Let $\phi \in C_0^\infty(\Omega^{**})$. We have:

$$\phi u = \phi U_p v = \phi U_p(\phi_1 v).$$

But $U_p(\phi_1 v) \in H^{s+a}$.

9. Regularity of formally hypoelliptic operators. Let $P(D)$ be a differential operator with constant coefficients on R^n. Hörmander has proved (in [1]) that $P(D)$ is hypoelliptic if and only if there is a number $d > 0$

such that, for every multi-index $\alpha = (\alpha_1, \cdots, \alpha_r)$ $(1 \leqq \alpha_j \leqq n)$, $|\alpha| \geqq 1$, and for every $\xi \in R_n$,

$$(1 + |\xi|^2)^{d/2} |P^{(\alpha)}(\xi)| \leqq C(1 + |P(\xi)|) \tag{9.1}$$

where $C < +\infty$ is independent of ξ.

Let Ω_1 be an open subset of a C^∞ manifold Ω covered by local coordinates $x^1, \cdots, x^n$. A differential operator $\boldsymbol{P}$ on Ω is said to be *formally hypoelliptic* in the local map $(\Omega_1, x^1, \cdots, x^n)$ if its expression $P(x, D)$ in the coordinates x^j satisfies the two following conditions:

$(FHE)_I$ *For every* $x_0 \in \Omega_1$, *the tangent differential operator* $P(x_0, D)$ (which has constant coefficients) *is hypoelliptic.*

$(FHE)_{II}$ *Whatever be the points* $x', x'' \in \Omega_1$, *there is a constant* $c > 0$ (depending on x', x'') *such that, for every* $\xi \in R_n$,

$$c \leqq \frac{1 + |P(x', \xi)|}{1 + |P(x'', \xi)|} \leqq \frac{1}{c}.$$

Several proofs of the hypoellipiticity of the formally hypoelliptic operators have been given (first by Hörmander [2] and Malgrange [1]; see also Peetre [1], Treves [1]). We shall indicate briefly the demonstration of the following fact:

THEOREM 2. *Let us assume that each point of* Ω *belongs to a local map in which* $\boldsymbol{P}$ *is formally hypoelliptic. Then* $\boldsymbol{P}$ *is regular in* Ω.

Since the question is of local nature, we may reason in a neighborhood of the origin in R^n (coordinates: $x^1, \cdots, x^n$). We may write

$$P(x, D) = P(0, D) + \sum_{j=1}^{N} a_j(x) P_j(D)$$

where, for every $j = 1, \cdots, N$, $a_j(x)$ is a C^∞ function in a neighborhood of 0, $a_j(0) = 0$, and $P_j(D)$ is hypoelliptic and satisfies

$$|P_j(\xi)| \leqq C(1 + |P(0, \xi)|), \qquad \xi \in R_n.$$

By hypothesis $P(0, D)$ is hypoelliptic. Now, whatever be s real, there is an

open neighborhood O_s of 0 such that $\phi \in C_0^\infty(O_s)$ implies, whatever be the multi-index α of length $|\alpha| \geqq 1$,

$$\| P^{(\alpha)}(0,D)\phi \|_{s+d_0} \leqq C \| P(0,D)\phi \|_s; \tag{9.2}$$

$$\| P_j^{(\alpha)}(D)\phi \|_{s+d_j} \leqq C \| P_j(D)\phi \|_s \leqq C' \| P(0,D)\phi \|_s, \tag{9.3}$$

where $d_0 > 0$, $d_j > 0$. Furthermore if $s \geqq 0$, we will have, for those ϕ,

$$\| P(0,D)\phi \|_s \leqq C \| \boldsymbol{P}\phi \|_s. \tag{9.4}$$

For the proofs of these statements, see Malgrange [1].

There is a multi-index α such that $P^{(\alpha)}(0,\xi)$ is a constant $\neq 0$. If the order of $P(0,D)$ is $\geqq 1$ (which we may of course assume), $|\alpha| \geqq 1$ and by combining (9.2) and (9.4), we get, at the origin and for $s \geqq 0$,

$$\| \phi \|_{s+d_0} \leqq C \| \boldsymbol{P}\phi \|_s.$$

This proves (A_s) for $s \geqq 0$ and $a = d_0$.

In order to prove (B_s), it is enough to consider the case where $\boldsymbol{L}$ is of order $\mu = 0$, i.e. $\boldsymbol{L}$ is the multiplication by a C^∞ function ψ defined near ω, and the case where $\boldsymbol{L} = \dfrac{\partial}{\partial x^\lambda}$ for a certain λ, $1 \leqq \lambda \leqq n$.

We have

$$[\boldsymbol{P}, \psi] = \sum_{|\alpha|>0} \frac{D_\alpha \psi}{|\alpha|!}\{P^{(\alpha)}(0,D) + \sum_{j=1}^{N} a_j(x) P_j^{(\alpha)}(D)\}.$$

We may apply (9.2), (9.3). We get, at the origin

$$\| [\boldsymbol{P}, \psi]\phi \|_{s+d} \leqq C \| P(0,D)\phi \|_s$$

with $d = \inf(d_0, d_1, \cdots, d_N)$.

Then (9.4) yields, for $s \geqq 0$

$$\| [\boldsymbol{P}, \psi]\phi \|_{s+d} \leqq C \| \boldsymbol{P}\phi \|_s.$$

Last

$$[\boldsymbol{P}, \frac{\partial}{\partial x^\lambda}] = \sum_{j=1}^{N} \frac{\partial a_j}{\partial x^\lambda}(x) P_j(D).$$

We apply successively (9.3) and (9.4). For $s \geqq 0$,

$$\| [\boldsymbol{P}, \frac{\partial}{\partial x^\lambda}] \phi \|_{s+1} \leqq C \| \boldsymbol{P} \phi \|_{s+1}$$

is valid at the origin.

This proves (B_s) for $s \geqq 0$ and $b = \inf(d, 1)$ (actually, it is obvious that d *is* always $\leqq 1$; $d = 1$ if and only if $\boldsymbol{P}$ is elliptic).

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PARTIALLY ANALYTIC SPACES.*

By Errett Bishop.

1. Introduction. The concept of a partially analytic space was introduced in a previous paper [1]. The present paper will refine and extend the theory. To this end it will be necessary to modify the definition of a partially analytic space which was given in [1]. Although formally quite different, the modified definition contains only one essentially new feature—local connectedness is no longer required.

Definition 1. *A partially analytic space* consists of a separable locally compact Hausdorff space K and a subalgebra $\mathfrak{A}$ of $C(K)$, where $C(K)$ is the algebra of all continuous complex-valued functions on K. The algebra $\mathfrak{A}$ contains the constant functions and is closed in the topology of uniform convergence on compact subsets of K. $\mathfrak{A}$ will be called the set of *analytic functions* on K.

In the previous paper certain structure manifolds were also postulated in the definition of a partially analytic space. The structure manifolds are now permitted to occur willy-nilly.

Definition 2. A *structure manifold* M of a partially analytic space K is a subset of K which has the structure of a complex analytic manifold such that (a) the manifold topology and the subset-of-K topology for M are the same, and (b) all functions in $\mathfrak{A}$ are analytic on M.

By *manifold* we mean connected separable complex analytic manifold.

Our first task is to show that the results of [1] remain valid for the more generally defined partially analytic spaces considered in this paper. This can be done by simple modifications of the proofs given in [1]. Proofs are only sketched. After this has been done we study a special kind of partially analytic space K, called an *embryonic space,* which is shown to have the property that the image of K under a proper analytic map of K into a manifold M is an analytic subset of M. This property of embryonic spaces is then applied to obtain simple proofs of two known results, (i) and (ii) below.

(i). Let A be an analytic set of dimension k in a manifold M. Let B be an analytic set in the manifold $M - A$ whose dimension at each point of

* Received May 11, 1961.

7

$M - A$ is at least $k+1$. We show that $\bar{B}$ is an embryonic space. Since the identity map of $\bar{B}$ into M is proper, we obtain as a corollary of the proper mapping theorem for embryonic spaces the Remmert-Stein theorem that $\bar{B}$ is an analytic subset of M. For this theorem see [9].

(ii). Let K be a holomorphically convex analytic space and $\mathfrak{A}$ a closed algebra of analytic functions on K with respect to which K is holomorphically convex. It is easy to see that the image K_0 of K under the natural map of K into $C^{\mathfrak{A}}$ (C denotes the complex numbers) is a separable locally compact Hausdorff space. The algebra $\mathfrak{A}$ can be considered as an algebra $\mathfrak{A}_0$ of functions on K_0, and together $\mathfrak{A}_0$ and K_0 constitute a partially analytic space. We show that this space is embryonic. From this fact we obtain Remmert's proper mapping theorem [8], which states that the image of an analytic space under a proper analytic map is an analytic subset of the image space.

If p is a point in a partially analytic space K, an n-tuple $f_1, \cdots, f_n$ of functions in $\mathfrak{A}$ will be said to be coordinates at the point p if there exists a neighborhood U of p which $f = (f_1, \cdots, f_n)$ maps properly into an open set $V \subset C^n$ such that for each h in $\mathfrak{A}$ there exists g analytic on V with $h/U = g \circ f$. Then K will be said to have rank at most n at p. We show that an embryonic space admits a set of coordinate functions at each point. It follows that an embryonic space can be canonically given the structure of an analytic space in the sense of Serre. Using this fact and the map described above of a holomorphically convex analytic space K onto a canonically associated embryonic space K we show that if $\mathfrak{A}$ is any closed algebra of analytic functions on K with respect to which K is holomorphically convex then there exists a canonical map ϕ of K onto an analytic space K_0 such that if $\mathfrak{A}_0$ denotes the set of functions f on K_0 such that $f \circ \phi \in \mathfrak{A}$ and if $\mathfrak{A}_{K_0}$ denotes the set of analytic functions on K_0 then $\mathfrak{A}_0 = \mathfrak{A}_{K_0}$. This theorem extends work of Stein [10], and has also been proved by Cartan [4], using other methods.

The theory of the imbedding of analytic spaces has been studied by Remmert, Narasimhan [7], and the author [1]. In this paper we study (Theorem 11) the problem of finding a proper analytic map of a holomorphically convex embryonic space K into complex Euclidean space C^N of sufficiently high dimension N which separates certain set of points of K and which locally generates all analytic functions at certain points of K. The imbedding theorems of [1] and [7] then follow as corollaries. The methods used for these constructions are basically those of [1] but also have something in common with the methods of [7].

The author is indebted to Hugo Rossi for many discussions about the

topics of this paper. Rossi has proved by different methods some results which are closely related to some of the results given below, in particular to Theorem 6 and to Corollary 2 of Theorem 8.

2. Partially analytic spaces. Since a new definition has been given of partially analytic spaces it becomes necessary to redefine analytic polyhedra.

DEFINITION 3. A subset L of a partially analytic space K is called a *frame* if it is contained in the union of some countable family of structure manifolds of K. The *dimension* of L is the least integer d (possibly $d=\infty$) such that L is contained in the union of a countable set of structure manifolds on each of which $\mathfrak{A}$ has dimension at most d. Let $f_1,\cdots,f_n$ functions in $\mathfrak{A}$ and let L be a frame. Write

$$P_0=\{p\colon p\in L, |f_i(p)|<1, 1\leqq i\leqq n\}.$$

Let P be a subset of P_0 which is both relatively open and relatively closed in P_0. If the closure $\bar{P}$ of P is a compact subset of L then P is called an *analytic polyhedron* with frame L defined by the functions $f_1,\cdots,f_n$. The *dimension* of P is the dimension of P when P itself is considered as a frame of K. P is called *reduced* if $n\leqq \dim P$, otherwise unreduced. An unreduced analytic polyhedron P is called *prepared* if it can be covered by a countable family $\{M_\alpha\}$ of structure manifolds such that for each α

(a) The dimension of $\mathfrak{A}$ on M_α does not exceed $\dim P$,

(b) For each α the algebra $\mathfrak{A}$ has dimension 0 on M_α over the set $(f_2f_1^{-1},\cdots,f_nf_1^{-1})$.

Bdry P denotes the boundary of P when considered as a subset of its frame L.

The proof of the following theorem is the same as the proof of Theorem 3 of [1], except for one modification which will be discussed below.

THEOREM 1. *Let $\{L_j\}$ be a finite family of frames in the partially analytic space K. Let the functions $f_1,\cdots,f_n$ in $\mathfrak{A}$ define for each j an analytic polyhedron P_j with frame L_j. Let $\gamma=\gamma(j)\leqq n$ be the dimensions of P_j. Let S_j be a compact subset of P_j. Then there exist functions $f_1',\cdots,f_n'$ in $\mathfrak{A}$, which can be taken to be polynomial functions of the functions $f_1,\cdots,f_n$, such that for each j the functions $f_1',\cdots,f_\gamma'$ define an analytic polyhedron P_j' with frame L_j such that $S_j\subset P_j'\subset P_j$. If $|f_i|<1$ for $1\leqq i\leqq n$ at all points of a compact set $T\subset K$ then the functions f_i' may be taken to have the same property.*

Remarks on the Proof of Theorem 1. We need only consider the one place at which the proof of Theorem 3 of [1] needs to be modified. The modification is necessary because we have dropped the local-connectedness condition when defining the frame L of an analytic polyhedron P. Because of the absence of local connectedness for L, a union of components of P_0 need no longer be an analytic polyhedron. This means that the proof of Theorem 2 of [1] must be modified. Theorem 2 of [1] states, in part, that if a prepared analytic polyhedron P is defined by functions $f_1, \cdots, f_n$, if $S \subset P$ is compact, and if $r > 1$ has the property that

$$S \subset P \cap \{p \colon |f_i(p)| < r^{-1}, 1 \leqq i \leqq n\},$$

then for N suppficiently large the functions

$$(rf_i(p))^N - (rf_1(p))^N \qquad 2 \leqq i \leqq n$$

define an analytic polyhedron Q with $S \subset Q \subset P$. To see this write

$$Q_0 = \{p \colon p \in L, |(rf_i(p))^N - (rf_1(p))^N| < 1, \quad 2 \leqq i \leqq n\}.$$

In [1] it was sufficient to take Q to be the union of those components of Q_0 which intersect S, but this need not give an analytic polyhedron if L is not locally connected. Thus a different construction of Q is needed. Write $Q_1 = \bar{Q}_0 \cap P$. As in [1] it follows that if N is sufficiently large then S and $(\text{bdry}\, P) \cap Q_1$ are disjoint closed subsets of the compact Hausdorff space Q_1 which are simultaneously intersected by no component of Q_1. It follows (see [6], p. 158) that there exists an open and closed subset Q_2 of Q_1 which contains S and is disjoint from bdry P. Thus $Q = Q_2 \cap Q_0$ is an open and closed subset of $Q_0 \cap P$ which is disjoint from bdry of P. It follows that $S \subset Q \subset P$ and that Q is open and closed in Q_0. Thus Q is the desired analytic polyhedron. Thus Theorem 2 of [1] can be established even when L is not locally connected. The rest of the proof of Theorem 3 of [1] is the same. Theorem 3 of [1] is just Theorem 1 above, except for the remark that the f_j' can be taken to be polynomials in the f_i. To see this note that the theory holds if $\mathfrak{A}$ is replaced by the closed subalgebra of $\mathfrak{A}$ generated by the functions $f_1, \cdots, f_n$. This implies that the functions f_j' can be taken to be uniform limits on compact sets of polynomials in the functions f_i, from which it easily follows that they can be taken to be polynomials in the f_i. Finally, it is clear from the proof of Theorem 3 of [1] that the f_i' can be chosen to have absolute values less than 1 at all points of T.

The following definitions and theorem correspond to Definitions 6 and 7 and Theorem 4 of [1]. The proof is the same.

DEFINITION 4. A partially analytic space K is called *weakly holomorphically convex* if K itself is a frame and if for each compact set $S \subset K$ each component of the set

$$\hat{S} = \{p \in K : |f(p)| \leqq \max_{q \in S} |f(q)|, \text{ all } f \text{ in } \mathfrak{A}\}$$

is compact.

DEFINITION 5. A continuous mapping ϕ of a topological space T_1 into a topological space T_2 is *almost proper* if each component of $\phi^{-1}(S)$ is compact for each compact set $S \subset T_2$.

THEOREM 2. *Let $\{L_j\}$ be a finite family of closed frames of a weakly holomorphically convex partially analytic space K, with $L_1 = K$. Let the dimension $\gamma = \gamma(j)$ of L_j be finite for each j. Let $n = \gamma(1)$ be the dimension of K. Then the set of elements $f = (f_1, \cdots, f_n)$ of the n-fold Cartesian product $\mathfrak{A}^n$ of $\mathfrak{A}$ such that for each j the mapping*

$$f^j : p \to (f_1(p), \cdots, f_\gamma(p))$$

of L_j into C^γ is almost proper is dense in $\mathfrak{A}^n$.

3. Special frames. Definition 6 which follows replaces Definition 8 of [1].

DEFINITION 6. A continuous proper map f from a subset S of a partially analytic space K to an n-dimensional complex analytic manifold M is called *special of dimension n* if all of the level sets of f on S are countable and if there exists a countable family $\{M_\alpha\}$ of n-dimensional structure manifolds of K which are subsets of S such that

(α) f is analytic on each M_α and non-singular on M_α at every point of M_α,

(β) Each M_α is open and $\cup M_\alpha$ is dense in S,

(γ) The set $H = f(S - \cup M_\alpha)$ is a countable union of locally analytic subsets of M, each of dimension at most $n - 1$.

It is worth noting that condition (α) may be replaced by the weaker condition

(α') f is analytic on each M_α and non-singular on M_α at *some* point of M_α.

To see this equivalence it is sufficient to note that if $\{M_\alpha\}$ satisfies (α'), (β),

and (γ) and if A_α denotes the singular set of f on M_α then the family $\{M_\alpha'\}$, where $M_\alpha' = M_\alpha - A_\alpha$, satisfies (α), (β), and (γ). The only condition that is not obvious is (γ), and (γ) follows from the inclusion

$$f(S - \cup M_\alpha') \subset \cup f(A_\alpha) \cup f(S - \cup M_\alpha)$$

and the fact that $f(A_\alpha)$ is a countable union of locally analytic sets of dimensions at most $n-1$.

Similarly we see that if the mapping f admits an extension to K such that $S - \cup M_\alpha$ is contained in the union of a countable family of structure manifolds of K of dimensions at most $n-1$ on each of which f is analytic then f satisfies condition (γ).

Before stating the next theorem we recall from [1] that if X is a topological space then ${}_kX$ denotes the k-fold unordered product of X, consisting of all unordered k-tuples of elements of X with the natural topology.

THEOREM 3. *Let f be a special map from a subset S of a partially analytic space K into a complex analytic manifold M, and let $\{M_\alpha\}$ be a countable family of n-dimensional structure manifolds satisfiyng (α), (β), and (γ). Let $G = \cup M_\alpha$. Then $H = f(S - G)$ is closed and nowhere dense in M, and $M - H$ is connected. There exists an integer $\lambda \geqq 1$, called the multiplicity of the map f, such that for each point z in $M - H$ there exist exactly λ distinct points p in G with $f(p) = z$. The map ω from $M - H$ into ${}_\lambda S$ which takes z into the unordered λ-tuple of such points can be uniquely extended to a continuous map ω of M into ${}_\lambda S$.*

Proof. The proof is the same as the proof of Theorem 5 of [1], except where the proof of said theorem is incomplete. In the penultimate paragraph of the proof of Theorem 5 of [1] it is shown that ω can be continuously extended to $(E^n - H) \cup \{z_0\}$ for each z_0 in H. To do this it is assumed that there are only a finite number of points p in $\bar{P}$ with $f(p) = z_0$. This fact is not evident without further consideration. Assume therefore that the set Γ of points in $\bar{P}$ mapping into z_0 is infinite. Since Γ is a countable compact set in $\bar{P}$, we see by the Baire category theorem that Γ has an infinite number of isolated points. Choose points $p_1, \cdots, p_{\lambda+1}$ which are isolated in Γ, and for $1 \leqq i \leqq \lambda + 1$ let U_i be a neighborhood of p_i with $\bar{U}_i \cap \Gamma = p_i$ such that the sets U_i are disjoint. As in the proof of Theorem 5 of [1] we see that there exists a neighborhood V of z_0 such that $f^{-1}(V) \cap \operatorname{bdry} U_i$ is void for all i. For each z in $V - H$ let $\delta_i(z)$, $1 \leqq i \leqq \lambda + 1$, be the number of points in $\omega(z) \cap U_i$. Since $V - H$ is connected we see that δ_i is constant on

$V - H$. Also $\delta_i \neq 0$ because if p is any point in $P - f^{-1}(H)$ sufficiently near to p_i then $f(p) = z \in U_i$ so that $\delta_i(z) \neq 0$. Thus

$$\lambda \geqq \sum \delta_i \geqq \lambda + 1,$$

a contradiction. Thus Γ is finite, as was to be proved.

COROLLARY. *If f is a special map from a subset S of a partially analytic space K into a complex analytic manifold M then the family $\{M_\alpha\}$ of Definition 6 can be taken to be a finite family of at most λ manifolds, such that $f(M_\alpha) \subset M - H$ for all α.*

Proof. Let A be the set of points p in M such that $\omega(p)$ consists of λ distinct points, so that A is open and dense in M. Because the complement of A has topological dimension at most $2n - 2$, A is connected. Now $f^{-1}(A)$ is a λ-sheeted covering space of A. Since A is connected, the set $\{M_\alpha'\}$ of components of $f^{-1}(A)$ is finite. It is then clear that each M_α' has the structure of a complex analytic manifold and that $\{M_\alpha'\}$ satisfies conditions (α), (β), and (γ) of Definition 6.

Corresponding to Lemma 5 of [1] we have the following theorem and corollary

THEOREM 4. *Under the assumptions of Theorem 3, let h be a continuous function on M which is analytic on $M - H$. Then h is analytic on M.*

Proof. The proof is essentially contained in the proof of Lemma 5 of [1]. H. Rossi suggests the following simple proof of Theorem 4 by induction on the dimension n of M. If $n = 1$ then H is a countable closed subset of M, so that the set H_0 of points in M where f is not analytic is countable and closed. Thus H_0 is void since otherwise H_0 would contain isolated points, contradicting the fact that a continuous function does not have isolated points of non-analyticity. This proves the theorem for $n = 1$. Consider $n > 1$ and assume the theorem true for $n - 1$. We may assume that M is $E^n = \{(z_1, \cdots, z_n) : |z_i| < 1, 1 \leqq i \leqq n\}$. Thus the intersection of H with all but a countable set of the varieties

$$L_c = \{(z_1, \cdots, z_n) \in E^n : z_1 = c\}$$

is a countable union of locally analytic subsets of L_c, each of dimension at most $n - 2$. By the induction hypothesis, f is analytic on all such L_c. Thus f is analytic on M.

COROLLARY. *Under the assumptions of Theorem 3, let g be a continuous*

function on S which is analytic on each M_α and let s be a symmetric polynomial in variables $z_1,\cdots,z_\lambda$. For each z in M let $p_1,\cdots,p_\lambda$ denote the points of $\omega(z)$, counted according to multiplicity. Then the function h on M defined by

$$h(z)=s(g(p_1),\cdots,g(p_\lambda))$$

is analytic on M.

Proof. By Theorem 4, it is sufficient to show that h is analytic at each point z_0 of $M-H$. Let $p_1{}^0,\cdots,p_\lambda{}^0$ be the points of $\omega(z_0)$. Thus z_0 has a neighborhood U such that there exist analytic homeomorphisms $\phi_1,\cdots,\phi_\lambda$ of z_0 onto neighborhoods of $p_1{}^0,\cdots,p_\lambda{}^0$ respectively with $\phi_i(z)\in\omega(z)$ for all z in U. Thus for all z in U we have

$$h(z)=s(g(\phi_1(z)),\cdots,g(\phi_\lambda(z))),$$

so that h is analytic at z_0, as was to be proved.

We now study conditions under which the image of a partially analytic space under proper maps will be an analytic subset of the image space.

LEMMA 1. *Let $f=(f_1,\cdots,f_n)$ be a special map of a subset S of a partially analytic space K into an n-dimensional manifold M. Let g be a continuous function on S with values in C^k which is analytic on each M_α, with $\{M_\alpha\}$ as in Definition 6. Let h be the map (f,g) of S into the manifold $\tilde{M}=M\times C^k$. Then $h(S)$ is an analytic subset of $\tilde{M}$.*

Proof. Consider a point r in $\tilde{M}-h(S)$, so that $r=(q,t_1,\cdots,t_k)$ where $q\in M$ and $t_1,\cdots,t_k$ are complex numbers. Let $q_1,\cdots,q_\lambda$ be the points in S which map into q under f, i.e., the points of $\omega(q)$ counted according to multiplicity. Thus for each i, $1\leqq i\leqq\lambda$, there exists $j=j(i)$, $1\leqq j\leqq k$, with $g_j(q_i)\neq t_j$. There thus exists a polynomial Δ_i in one variable such that $\Delta_i(t_j)=1$ and $\Delta_i(g_j(q_i))=0$. Define the analytic function Δ on C^k by

$$\Delta(z_1,\cdots,z_k)=\prod_{i=1}^{\lambda}\Delta_i(z_{j(i)}).$$

Thus $\Delta(t_1,\cdots,t_k)=1$ and $\Delta(g(q_i))=0$ for $1\leqq i\leqq\lambda$. It follows that $\boldsymbol{\Delta}=\Delta\circ g$ is a continuous function on S which is analytic on each M_α, with $\boldsymbol{\Delta}(q_i)=0$ for $1\leqq i\leqq\lambda$. If for each p in M we let $p_1,\cdots,p_\lambda$ denote the points of $\omega(p)$ then by the corollary of Theorem 4 we seee that each of the elementary symmetric functions of the quantities $\boldsymbol{\Delta}(p_1),\cdots,\boldsymbol{\Delta}(p_\lambda)$ is an analytic function of the point p *in* M. Thus the function F on $\tilde{M}$ defined by

$$F(p,z_1,\cdots,z_k)=\prod_{i=1}^{\lambda}(\Delta(z_1,\cdots,z_k)-\boldsymbol{\Delta}(p_i))$$

is analytic on M. Clearly

$$F(r) = F(q, t_1, \cdots, t_k) = \prod_{i=1}^{\lambda} (1 - \Delta(q_i)) = 1.$$

Also if $(p, z_1, \cdots, z_k)$ is in $h(S)$ then there exists i, $1 \leqq i \leqq \lambda$, such that $z_1 = g_1(p_i), \cdots, z_k = g_k(p_i)$. Hence $F(p, z_1, \cdots, z_k)$ has the factor $\Delta(z_1, \cdots, z_k) - \Delta(p_i) = \Delta(z_1, \cdots, z_k) - \Delta(g_1(p_i), \cdots, g_k(p_i)) = 0$. Thus F vanishes on $h(S)$ and $F(r) = 1$. Therefore $h(S)$ is an analytic set, as was to be proved.

DEFINITION 7. An n-dimensional frame L in a partially analytic space K is called *special* if there exists a countable family $\{M_\alpha\}$ of n-dimensional structure manifolds of K such that

(a) $M_\alpha \subset L$ for all α,

(b) Each M_α is an open set in L,

(c) The dimension of the frame $L - \cup M_\alpha$ is at most $n - 1$,

(d) $\cup M_\alpha$ is dense in L,

(e) There exists a set $N \supset L$ such that the level sets of $\mathfrak{A}$ on N are all countable and such that N is a countable union of structure manifolds of K.

The family $\{M_\alpha\}$ is called a *structural family* for the frame L.

It is clear that if L is a special frame of dimension n in a partially analytic space K then a proper map $f = (f_1, \cdots, f_n)$ with $f_i \in \mathfrak{A}$, $1 \leqq i \leqq n$, from L into an open subset of C^n whose level sets on L are all countable is special whenever L has a structural family $\{M_\alpha\}$ such that f is non-singular at some point of each M_α.

LEMMA 2. *Let $g = (g_1, \cdots, g_k)$, with $g_i \in \mathfrak{A}$, $1 \leqq i \leqq k$, be a proper map of an n-dimensional special frame L in a partially analytic space K into an open subset U of C^k. Then $g^{-1}(p_0) \cap L$ is finite for all p_0 in U.*

Proof. Consider a point p_0 in U. Since g is proper, there exists an analytic polyhedron Q with frame L defined by functions which are linear combinations of the functions $g_1, \cdots, g_k$ such that $g^{-1}(p_0) \subset Q$. Since the dimension of $\mathfrak{A}$ on L is n, there exist functions $f_1, \cdots, f_n$ in $\mathfrak{A}$ defining an analytic polyhedron P with $g^{-1}(p_0) \subset P$. By changing the functions $f_1, \cdots, f_n$ slightly we may assume, by Theorem 1 of [1], that $f_1, \cdots, f_n$ have countable level sets on N. Thus f is a special map of P into E^n. If we let h be the map (f, g) of P into $E^n \times U$, we see by Lemma 1 that $h(P)$ is an analytic subset of $E^n \times U$. Thus

$$T = h(P) \cap (E^n \times \{p_0\}) = f(g^{-1}(p_0)) \times \{p_0\}$$

is a compact analytic subset of $E^n \times U \subset C^{n+k}$. Therefore T is finite. Thus $f(g^{-1}(p_0))$ is finite. Since f is at most λ to one, $g^{-1}(p_0)$ is finite. This proves the lemma.

THEOREM 5. *If $g = (g_1, \cdots, g_k)$ is a proper map of an n-dimensional special frame L in a partially analytic space K into an open set $U \subset C^k$, where $g_i \in \mathfrak{A}$ for all i, then $g(L)$ is an analytic subset of U.*

Proof. Since g is proper, $g(L)$ is closed in U. Thus we need only check that $g(L)$ is analytic at each point p_0 in $g(L)$. Let $\mathfrak{A}_0$ be the closure in the topology of uniform convergence on compact subsets of K of the set of all polynomials in the functions $g_1, \cdots, g_k$, so that $(K, \mathfrak{A}_0)$ is a partially analytic space. Let V be a small neighborhood of p_0 with $\bar{V} \subset U$. Since the map g is proper there exists an analytic polyhedron Q in $(K, \mathfrak{A}_0)$ with frame L with $L \cap g^{-1}(V) \subset Q$. By Theorem 1 there exists an analytic polyhedron P defined by functions $f_1, \cdots, f_n$ in $\mathfrak{A}_0$ with $L \cap g^{-1}(V) \subset P \subset Q$. Thus f can be changed slightly such that each f_i is a polynomial function of $g_1, \cdots, g_k$. Thus f is a proper map from an analytic polyhedron $P \supset L \cap g^{-1}(V)$ into E^n, such that there exist polynomial functions $\Delta_1, \cdots, \Delta_n$ on C^k with

$$f_i(p) = \Delta_i(g_1(p), \cdots, g_k(p))$$

for all p in K. Since P is an open subset of the special frame L it is itself a special frame, so that by Lemma 2 f is a special map of P into E^m. By Lemma 1 we see that $(f, g)(P) = A$ is an analytic set in $E^n \times U$, which clearly consists of all $(\Delta(p), p)$ for $p \in g(P)$, where $\Delta = (\Delta_1, \cdots, \Delta_n)$. Let $\gamma_1, \cdots, \gamma_j$ be analytic functions defined on some neighborhood Ω in $E^n \times U$ of the point $(\Delta(p_0), p_0)$ whose set of common zeros is $A \cap \Omega$. Choose a neighborhood W of p_0 with $W \subset V$ such that $(\Delta(p), p) \in \Omega$ for all p in W. Let B be the set of zeros on W of the analytic functions $\gamma_1(\Delta(p), p), \cdots, \gamma_j(\Delta(p), p)$. If $p \in B$ then $(\Delta(p), p) \in A$ so that $p \in g(P) \subset g(L)$. If $p \in g(L) \cap W = g(P) \cap W$ then $(\Delta(p), p) \in A \cap \Omega$ so that $\gamma_1(\Delta(p), p) = \cdots = \gamma_j(\Delta(p), p) = 0$ and therefore $p \in B$. Thus $g(L) \cap W = B$ so $g(L)$ is analytic at the point p_0, as was to be proved.

DEFINITION 8. A frame L in a partially analytic space K which is the union of a countable family of closed special frames only finitely many of which intersect any compact subset of L is called an *embryonic frame.* If K itself is an embryonic frame then K is called an *embryonic space.*

For embryonic frames we have the following immediate corollary of Theorem 5.

COROLLARY. *If L is an embryonic frame and $f_1, \cdots, f_k$ in $\mathfrak{A}$ define a*

proper map of L into an open set U in C^k then $f(L)$ is an analytic subset of U.

As a corollary of this corollary, we obtain the following theorem.

THEOREM OF REMMERT-STEIN. *Let A be a k-dimensional analytic subset of an open set $U \subset C^n$. Let B be an analytic set in $U - A$ of dimension at least $k+1$ at each of its points. Then $\bar{B} \cap U$ is an analytic subset of U.*

Proof. Let $K = A \cup B$ and let $\mathfrak{A}$ be the closure on K in the topology of uniform convergence on compact sets of all functions analytic in U. Thus $(K, \mathfrak{A})$ is a partially analytic space. Clearly $L = \bar{B} \cap U$ is an embryonic frame in K. Let f be the inclusion map of L in U. By the above corollary, $\bar{B} = f(L)$ is an analytic subset of U, as was to be proved.

4. Contractions of analytic spaces. The term *analytic space* will be used in the sense of Serre (see [5], p. 280).

DEFINITION 9. Let K be an analytic space and $\mathfrak{A}$ be a closed algebra of functions everywhere analytic on K. Assume that K is holomorphically convex in the sense that the set $\tilde{S}$ of Definition 4 is compact for all compact $S \subset K$. Let K_0 be the image of K under the natural map ϕ of K into $C^{\mathfrak{A}}$. Since K is holomorphically convex, the map ϕ is proper and K_0 is closed in $C^{\mathfrak{A}}$, so that K_0 is locally compact. Let $\mathfrak{A}_0$ be the subalgebra of $C(K_0)$ consisting of all f in $C(K_0)$ such that $f \circ \phi \in \mathfrak{A}$. It is clear that $(K_0, \mathfrak{A}_0)$ is a holomorphically convex partially analytic space which we call the *contraction* of $(K, \mathfrak{A})$.

Of course Definition 9 could be made for an arbitrary partially analytic space K, but we shall only apply the construction to analytic spaces. The object of this section will be to show that $(K_0, \mathfrak{A}_0)$ is actually an embryonic space.

LEMMA 3. *With the notation of Definition 9, let M be a structure manifold of K on which $\mathfrak{A}$ has dimension n. Then $\phi(M)$ is a frame in K_0 of dimension at most n.*

Proof. The proof is by induction on the dimension d of M. If $d = 0$ the lemma is clearly true, so assume that the lemma is proved for all d smaller than the actual dimension d of M. Choose $f_1, \cdots, f_n$ in $\mathfrak{A}$ to have rank n at some point of M, and let A be the analytic subset of M at which these functions have rank less than n. Thus A is a countable union of structure manifolds of K each of dimension less than d on each of which $\mathfrak{A}$ has rank at most n. By the induction hypothesis $\phi(A)$ is a frame of dimension at most n. On the other hand, $M - A$ can be covered by a countable family

$\{M_\alpha\}$ manifolds each of which is mapped by $f_1, \cdots, f_n$ onto a connected open set L_α in C^n and on each of which $f_1, \cdots, f_n$ and $\mathfrak{A}$ have the same level sets. Thus L_α is naturally homeomorphic to $\phi(M_\alpha)$; the $\phi(M_\alpha)$ are thus a countable family of structure manifolds of K_0 on each of which $\mathfrak{A}_0$ has dimension n. Thus $\phi(M - A)$ is also a frame of dimension at most n. This completes the proof.

LEMMA 4. *Let S be an analytic space whose set R of regular points is connected and let $\mathfrak{A}$ be an algebra of analytic functions on S of dimension at most n at each point of R. Let M be any structure manifold of S. Then $\mathfrak{A}$ has dimension at most n on M.*

Proof. If d is the dimension of S then every level set of $\mathfrak{A}$ on R has dimension at least $d - n$ at each point. Therefore, by a known result ([3], Exp. XIV), every level set of $\mathfrak{A}$ on S has dimension at least $d - n$ at each point. Assume that the dimension of $\mathfrak{A}$ on M is larger than n. Thus there exists $f = (f_1, \cdots, f_{n+1})$ in $\mathfrak{A}^{n+1}$ having rank $n + 1$ at some point of M, so that $f(M)$ contains an open set U of C^{n+1}. Now S can be covered by a countable family $\{M_\alpha\}$ of structure manifolds such that for each α either f has rank $n + 1$ at every point of M_α or at no point of M_α. Let $\{M_\alpha'\}$ be the subfamily of $\{M_\alpha\}$ consisting of those M_α on which f has rank less than $n + 1$. If p is any point in U then $f^{-1}(p)$ contains some level set of $\mathfrak{A}$ on S and therefore has dimension at least $d - n$. By the theorem on the dimension of a countable union of closed sets, $f^{-1}(p) \cap M_\alpha$ has dimension at least $d - n$ for some α. Since the dimension of M_α is at most d it follows that f does not have rank $n + 1$ at all points of M_α, so that M_α is some M_α'. Thus U is covered by the sets $f(M_\alpha')$. Since f has rank less than $n + 1$ at all points of M_α', $f(M_\alpha')$ is a countable union of closed sets of topological dimensions at most $2n$. Thus U is a countable union of closed sets of topological dimensions at most $2n$. This contradiction proves Lemma 4.

THEOREM 6. *With the notation of Definition 9, $(K_0, \mathfrak{A}_0)$ is an embryonic space.*

Proof. The analytic space K is the union of a countable family $\{M_\alpha\}$ of irreducible sets, only finitely many of which intersect any compact subset of K. Thus $\{\phi(M_\alpha)\}$ is a covering of K_0 by closed frames, only finitely many of which intersect any compact subset of K_0. Thus it is sufficient to show that each $\phi(M_\alpha)$ is an embryonic frame in K_0. It is therefore enough to show that $\phi(T)$ is an embryonic frame in K_0 for each irreducible analytic subset T of K. We will do this by an induction on the dimension d of T. The case $d = 0$ is clear. Thus we assume that the statement has been verified for

all T of dimension smaller than the dimension d of the given T. Let S be the set of singular points of T, so that $\dim S < d$. Since S is the union of of a countable family of irreducible analytic sets of dimensions less than d, only finitely many of which intersect any compact set, we see by the induction hypothesis that $\phi(S)$ is an embryonic frame in K_0. Thus if $\phi(S) = \phi(T)$ then $\phi(T)$ is an embryonic frame, as was to be proved. Therefore we may assume that $\phi(S) \neq \phi(T)$. We shall prove that the set $S_1 = T \cap \phi^{-1}(\phi(S))$ is an analytic subset of K. To this end, let p be any point in K, and let U be a neighborhood of p such that $\bar{U}$ is compact. Since K is holomorphically convex there exists an analytic polyhedron P with frame K_0 defined by functions $f_1, \cdots, f_n$ in $\mathfrak{A}_0$ such that $\phi(\bar{U}) \subset P$. Now let g be any element of $\mathfrak{A}^N$, where N is any positive integer and let $\hat{g}$ be the element of $\mathfrak{A}_0{}^N$ corresponding to g. Since $\phi(S)$ is an embryonic frame in K_0, $\phi(S) \cap P$ is an embryonic frame in the partially analytic space $(P, \mathfrak{A}')$ obtained by taking $\mathfrak{A}'$ to be all uniform limits on compact subsets of P of functions in $\mathfrak{A}_0$. By the corollary of Theorem 5 it follows that $(f_1, \cdots, f_n, \hat{g}) = \Delta$ maps $\phi(S) \cap P$ onto an analytic set B_g in $E^n \times C^N$. Hence the subset

$$U \cup (\Delta \circ \phi)^{-1}(B_g) = A_g = \{p \in U : (f_1(\phi(p)), \cdots, f_n(\phi(p)), g(p)) \in B_g\}$$

of U is analytic. Now clearly

$$U \cap \phi^{-1}(\phi(S)) = \bigcap A_g,$$

where the intersection is taken over all finite families g of functions in $\mathfrak{A}$. Thus $U \cap \phi^{-1}(\phi(S))$ is an analytic subset of U. It follows that $\phi^{-1}(\phi(S))$ is analytic subset of K. Thus $S_1 = T \cap \phi^{-1}(\phi(S))$ is an analytic set.

Now let k be the smallest of the local dimensions of all of the level sets of $\mathfrak{A}$ on T. By a known theorem ([3], Exp. XIV), the set J of p in T such that the level set of $\mathfrak{A}$ on T which passses through p has dimension k at p is an open set in T. Thus J intersects $T - S_1$ so that $\mathfrak{A}$ has rank $d - k$ at some point of $T - S_1$.

Now S_1 is the union of a countable family $\{M_\alpha\}$ of structure manifolds of K. Since the level sets of $\mathfrak{A}$ on S_1 have dimension at least k at each point, by Lemma 4 we see that $\mathfrak{A}$ is of dimension at most $d - 1 - k$ on each M_α. By Lemma 3 it follows that $\phi(S_1)$ is a frame in K_0 of dimension at most $d - k - 1$.

Now let S_2 be the set of all points in $T - S_2$ at which $\mathfrak{A}$ has rank $d - k - 1$ or less. Clearly S_2 is an analytic subset of $T - S_1$. Since $T - S_1$ is a (connected) manifold, S_2 is a proper analytic subset of $T - S_1$.

Now if $\mathfrak{A}$ had rank $d - k$ or more on S_2 at some regular point of S_2 then $\mathfrak{A}$ would have rank at least as great on $T - S_1$ at that point. Therefore

$\mathfrak{A}$ has rank at most $d-k-1$ at all regular points of S_2. Thus the dimensions of $\mathfrak{A}$ at every regular point of S_2 is at most $d-k-1$. By Lemma 4 it follows that S_2 can be covered by a countable family of structure manifolds on each of which $\mathfrak{A}$ has dimension at most $d-k-1$. Thus $\phi(S_2)$ is a frame in K_0 of dimension at most $d-k-1$. If $S_3=S_2\cup S_1$ it follows that the frame $\phi(S_3)$ in K_0 has dimension at most $d-k-1$. If $S_4=T\cap\phi^{-1}(\phi(S_3))$ it follows that $H=T-S_4$ is open and dense in T and that $\phi(S_4)=\phi(S_3)$. Since $\mathfrak{A}$ has rank $d-k$ at each point of H, the level sets of $\mathfrak{A}$ on H will all be k-dimensional submanifolds of H or countable unions of such submanifolds. Now since the level sets of $\mathfrak{A}$ on H are closed in T, and therefore compact, each such level set will actually be a finite union of compact k-dimensional submanifolds M of H. Clearly the set $\tilde{H}$ of all such M has the structure of a $d-k$ dimensional manifold, and $\mathfrak{A}$ gives rise to an algebra $\tilde{\mathfrak{A}}$ of analytic functions on $\tilde{H}$ whose level sets on $\tilde{H}$ are all finite. Also $\tilde{\mathfrak{A}}$ has rank $d-k$ at each point of $\tilde{H}$ because $\mathfrak{A}$ has rank $d-k$ at each point of H.

Now if p_1 and p_2 are distinct points in $\tilde{H}$ with $h(p_1)=h(p_2)$ for all h in $\tilde{\mathfrak{A}}$, we may choose $(f_1,\cdots,f_{d-k})=f$ in $\tilde{\mathfrak{A}}^{d-k}$ having rank $d-k$ at both p_1 and p_2, so that there exist neighborhoods U_1 and U_2 of p_1 and p_2 respectively each of which f maps homeomorphically onto the same open set $V\subset C^{d-k}$. Thus there exists a unique analytic homeomorphism σ of U_1 onto U_2 such that f and $f\circ\sigma$ agree on U_1. If $h\in\tilde{\mathfrak{A}}$ then the set

$$A_h=\{q\in U_1: h(q)=h(\sigma(q))\}$$

is an analytic set in U_1. Clearly $\bigcap A_h=A$ is the set of all q in U_1 which are identified with some point in U_2 by all functions in $\tilde{\mathfrak{A}}$. If $A=U_1$ we write $p_1\equiv p_2$. This clearly gives an equivalence relation on $\tilde{H}$, and the equivalence classes form a set $\hat{H}$ which has a canonical structure of a $d-k$ dimensional manifold in such a way that $\tilde{H}$ is a covering manifold of $\hat{H}$. Also $\tilde{\mathfrak{A}}$ gives rise to an algebra $\hat{\mathfrak{A}}$ of analytic functions on $\hat{H}$. If $p_1\neq p_2$ in $\hat{H}$ and $h(p_1)=h(p_2)$ for all h in $\hat{\mathfrak{A}}$, define B_h just as A_h was defined, with $\tilde{H}$ replaced by $\hat{H}$ and $\tilde{\mathfrak{A}}$ by $\hat{\mathfrak{A}}$. Thus B is an analytic set in some neighborhood of p_1 and consists of all points in that neighborhood which are identified by $\hat{\mathfrak{A}}$ with some point in a fixed neighborhood of p_2. Since for each p_1 the number of possible p_2 is finite, it follows that there exists a neighborhood U of p_1 such that all points in U which are identified with some other point of $\hat{H}$ by all functions in $\hat{\mathfrak{A}}$ are a proper analytic subset of U. Thus the set W of all points in $\hat{H}$ which are identified with some other point in $\hat{H}$ by all functions in $\hat{\mathfrak{A}}$ is a proper analytic subset of $\hat{H}$.

If γ denotes the natural map of H onto $\hat{H}$ it follows that $\gamma^{-1}(W)$ is a

proper analytic subset of H. As above we see that $\phi(\gamma^{-1}(W))$ is a frame in K_0 of dimension at most $d-k-1$. With $H_0 = H - \gamma^{-1}(W)$ it follows that $\phi(T-H_0) = \phi(\gamma^{-1}(W) \cup S_4)$ is a frame in K_0 of dimension at most $d-k-1$. Since H_0 is dense in T, $\phi(H_0)$ is dense in $\phi(T)$. Clearly $\phi(H_0)$ is in one-one correspondence with $\hat{H} - W$. It is easy to see that $\phi(H_0)$ and $\hat{H} - W$ are actually analytically homeomorphic in a natural way. Thus $\phi(H_0)$ is a $d-k$ dimensional structure manifold in K_0, at each point of which $\mathfrak{A}_0$ has rank $d-k$. Since $T-H_0$ is closed in T and therefore in K, $\phi(T-H_0)$ is closed in K_0. Since by construction $\phi(H_0)$ and $\phi(T-H_0)$ are disjoint, we see that $\phi(H_0)$ is open in the frame $\phi(T)$. It follows that $\phi(T)$ is a $d-k$ dimensional special frame in K_0, as was to be proved.

As a corollary, we obtain the proper mapping theorem of Remmert.

THEOREM. *If f is a proper analytic mapping of an analytic space K_1 into an analytic space K_2 then $f(K_1)$ is an analytic set in K_2.*

Proof. Since the analytic structure of K_2 is defined by local imbeddings of K_2 onto E^N for some N, there is no loss of generality in assuming that $K_2 = E^N$. Let K_0 be the contraction of $(K_1, \mathfrak{A})$, where $\mathfrak{A}$ is the algebra of all analytic functions on K_1. Since the coordinates of the mapping f are in $\mathfrak{A}$, we see that f gives rise to a proper analytic map $\bar{f}$ of K_0 into E^N, whose coordinates are in $\mathfrak{A}_0$. By Theorem 5 it follows that $f(K_1) = \bar{f}(K_0)$ is an analytic subset of E^N, as was to be proved.

5. Imbeddings and related maps.

DEFINITION 10. Let K be a partially analytic space. A set $S \subset \mathfrak{A}^N$ is *strictly of the first category* if it is contained in the union of a countable family $\{S_i\}$ of nowhere dense sets having the following property. For each i there exists a compact set $F_i \subset K$ and a closed subset J_i of $(C(F_i))^N$ such that S_i consists of all functions in $\mathfrak{A}^N$ whose restriction to F_i is in J_i.

THEOREM 7. *Let U and V each be a countable union of structure manifolds of dimensions at most n of a partially analytic space K. Let $f = (f_1, \cdots, f_n)$ in $\mathfrak{A}^n$ have countable level sets on $U \cup V$. For each p in U and q in V let there exist h in $\mathfrak{A}$ with $h(p) \neq h(q)$. Then the complement F of the set $G = G(U, V)$ of all g in $\mathfrak{A}^{n+1}$ for which $(f, g)(U)$ and $(f, g)(V)$ are disjoint is strictly of the first category in $\mathfrak{A}^{n+1}$.*

Proof. It is clearly enough to show that to each p in U and q in V there exist neighborhoods U' of p in U and V' of q in V such that the comcomplement of $G(U', V')$ is strictly of first category. This is clearly true unless $f(p) = f(q)$. Thus we may assume that U and V are mapped by f into the same open set Γ in C^n. Let $U = \cup U_k$, $V = \cup V_k$, where

$\{U_k\}$ and $\{V_k\}$ are countable families of compact sets. Let T_k consist of all g in $\mathfrak{A}^{n+1}$ such that $(f,g)(U_k)$ and $(f,g)(V_k)$ are disjoint. It is clearly sufficient to prove that for each k the complement S_k of T_k is strictly of first category. If we let J_k be the closure of S_k in $(C(U_k \cup V_k))^{n+1}$ it is sufficient to show that

(a) T_k is dense in $\mathfrak{A}^{n+1}$,

and

(b) For each g in T_k the restriction of g to $U_k \cup V_k$ is not in J_k.

We first prove (a). To this end, it is sufficient to show that the set T of all g in $\mathfrak{A}^{n+1}$ such that $(f,g)(U)$ and $(f,g)(V)$ are disjoint is dense. Consider therefore $g = (g_1, \cdots, g_{n+1})$ in $\mathfrak{A}^{n+1}$. Assume by induction on k that for each k, $0 \leqq k \leqq n+1$, there exist $g_1', \cdots, g_k'$ arbitrarily near to $g_1, \cdots, g_k$ respectively such that the set of points p in U with

$$(f(p), g_1'(p), \cdots, g_k'(p)) = (f(q), g_1'(q), \cdots, g_k'(q))$$

for some q in V is a frame L_k of dimension at most $n-k$, and the same for the corresponding frame M_k of V. This is clearly true if $k=0$. Thus by induction we may assume $g_1', \cdots, g_k'$ already chosen, $k < n+1$. The set of g_{k+1}' in $\mathfrak{A}$ such that for each frame L of a countable family of frames of dimensions at most $n-k$ covering L_k and each frame M of a countable family of frames of dimensions at most $n-k$ covering M_k there exists a dense countable set of points p in L and a dense countable set of points q in M such that

$$g_{k+1}'(p) \notin g_{k+1}'(V \cap f^{-1}(f(p)))$$

and

$$g_{k+1}'(q) \notin g_{k+1}'(U \cap f^{-1}(f(q)))$$

by obvious category arguments has a complement which is a set of first category in $\mathfrak{A}$. Thus we may find g_{k+1}' arbitrarily close to g_{k+1}. Now $(f, g_{k+1}')(M)$ is a countable union of locally analytic subsets of C^{n+k+1}, so that $A = L \cap (f, g_{k+1}')^{-1}(M)$ is a countable union of locally analytic subsets of L. Since by the choice of g_{k+1}' the set A has no interior in L, the frame A has dimension at most $n-k-1$. Since this is true for all L and M it follows that L_{k+1} and M_{k+1} are frames of dimensions at most $n-k-1$. This completes the induction. Letting $k = n+1$ we see that (a) is true.

To prove (b), notice that for each g in S_k the pair (f,g) identifies some point in U_k with some point in V_k. The same property therefore holds for J_k. From this (b) is clear and the lemma is proved.

Corollary. *If L is an n-dimensional frame, if $\mathfrak{A}$ separates points of L,*

and if $f \in \mathfrak{A}^n$ *has countable level sets on* L *then the set* T *of all* g *in* $\mathfrak{A}^{n+1}$ *such that* (f, g) *is one-to-one on* L *has a complement* S *which is strictly of first category in* $\mathfrak{A}^{n+1}$.

Proof. The set $L \times L - \{(q, q) : q \in L\}$ can be covered by a countable family $\{U_k \times V_k\}$, where U_k and V_k are disjoint countable unions of frames of dimensions at most n. By the theorem, the complement of the set of all g such that $(f, g)(U_k)$ and $(f, g)(V_k)$ are disjoint is strictly of first category in $\mathfrak{A}^{n+1}$. From this the corollary follows easily.

DEFINITION 11. A point p in a frame L in a partially analytic space K is called an *analytic point* of L of dimension n and rank at most N if there exists a neighborhood U in L of p and f in $\mathfrak{A}^N$ mapping U homeomorphically onto an n-dimensional analytic subset of an open set $S \subset C^N$ such that for each h in $\mathfrak{A}$ there exists an analytic function w on S such that

$$h(p) = w(f(p))$$

for all p in U. Then f is called a set of coordinates at p and U is the corresponding coordinate neighborhood.

THEOREM 8. *Every point* p *of an embryonic frame* L *is an analytic point of* L *if* $\mathfrak{A}$ *separates points of* K.

Proof. By the corollary just proved, there exist $f_1, \cdots, f_N$ in $\mathfrak{A}$ mapping some neighborhood V of p one-to-one and properly onto an analytic set $A \subset E^N$. Let H consist of all continuous functions h on A such that there exists g in $\mathfrak{A}$ with $g(q) = h(f(q))$ for all q in U, and let G be the set of germs at $z = f(p)$ of H. Let F be the set of germs of all functions bounded and analytic in some neighborhood of z on the regular points of A, so F consists of the germs at z of all functions analytic on the normalization of A (see [5]). Let R consist of all functions in some neighborhood of z on A which can be extended to be analytic in some E^N neighborhood of z. Thus $R \subset G \subset F$, and G and F are R-modules. By the remark at the top of p. 291 of [5], the R-module F is finitely generated. Since R is noetherian, it follows that the sub-module G of F is also finitely generated. Thus there exist $g_1, \cdots, g_k$ in $\mathfrak{A}$ such that for each g in $\mathfrak{A}$ there exist $h_1, \cdots, h_k$ in R with

$$(*) \qquad \tilde{g}(t) = \sum_{i=1}^{k} h_i(t)\tilde{g}_i(t)$$

for all t in $A \cap V_g$, where V_g is some neighborhood of z and $\tilde{g}$ is the element in G corresponding to g in $\mathfrak{A}$. Thus if $\{S_n\}$ is a decreasing sequence of neighborhoods of z whose intersection is z, we see that for each g in $\mathfrak{A}$ there exists $n = n(g)$ with $S_n \subset V_g$ and $(h) \leqq n$, where

$$(h) = \sup\{|h_i(t)| : 1 \leqq i \leqq k, t \in S_n\}$$

with h_i as above. Hence $\mathfrak{A} = \bigcup \mu_n$, where

$$\mu_n = \{g : n(g) = n\}.$$

Since each μ_n is closed in $\mathfrak{A}$, some μ_n contains an open subset of $\mathfrak{A}$, for otherwise $\mathfrak{A}$ would be of first category. Thus actually $S_n \subset V_g$ for all g in $\mathfrak{A}$ for the right choice of V_g. Let $S = S_n \times C^k$, so that $f_0 = (f, g_1, \cdots, g_k)$ maps $f^{-1}(S_n) \cap U$ properly onto an analytic subset of S. For each g in $\mathfrak{A}$ we have $(*)$, so that

$$g(q) = \sum_{i=1}^{k} h_i(f(q))\tilde{g}_i(f(q)) = \phi(f_0(q)),$$

for all q in $f^{-1}(S_n) \cap U$, where

$$\phi : (t, z_1, \cdots, z_k) \to \sum_{i=1}^{k} h_i(t) z_i$$

is analytic on S. This completes the proof.

Corollary 1. *Every embryonic space K for which $\mathfrak{A}$ separates points admits an analytic space structure K_0 so that all functions in $\mathfrak{A}$ are analytic on K_0. If K is holomorphically convex and if $\mathfrak{A}$ separates points of K then $\mathfrak{A}$ consists of all functions analytic on K_0.*

Proof. The first statement is an immediate consequence of Theorem 8. To prove the second statement, consider any h analytic on K_0. If S is any compact subset of K let P be an analytic polyhedron in K containing S defined by $f = (f_1, \cdots, f_n)$ in $\mathfrak{A}^n$. By Theorem 8, we may assume, by adding more functions if necessary, that $f_1, \cdots, f_n$ are a coordinate set of function at each point of P, and by the corollary to Theorem 7 we may assume that f separates points of P. Thus f maps P one-to-one onto an analytic set $A \subset E^n$. Let g be that function on A such that $h(p) = g(f(p))$ for all p in P. Since f is a coordinate system at each point of P and since h is analytic on K_0 the function g on A can be locally extended at each point of A to an analytic function in some neighborhood of that point. As is well known, it follows that g can be extended analytically to E^n. Thus g can be uniformly approximated on $f(S)$ by polynomials in the coordinate functions. It follows that h can be uniformly approximated on S by polynomials in the functions $f_1, \cdots, f_n$. Since S is any compact subset of K it follows that $h \in \mathfrak{A}$, as was to be proved. The author is indebted to Hugo Rossi for this argument.

Corollary 2. *Every holomorphically convex analytic space K admits an analytic proper map ϕ onto a holomorphically convex analytic space $\tilde{K}$ such that*

(a) *The algebra $\tilde{\mathfrak{A}}$ of analytic functions on $\tilde{K}$ separates point of $\tilde{K}$,*

(b) $\mathfrak{A} = \{f \circ \phi : f \in \tilde{\mathfrak{A}}\}$,

and

(c) *There are enough analytic functions in $\tilde{\mathfrak{A}}$ to give a coordinate set of functions at each point of $\tilde{K}$.*

Proof. This corollary is an immediate consequence of Theorem 6 and Corollary 1 of Theorem 8.

THEOREM 9. *Let p be an analytic point of dimension n and rank at most N in a frame L of a partially analytic space K, so that there exists $f=(f_1,\cdots,f_N)$ in $\mathfrak{A}^N$ forming a coordinate set of functions on some neighborhood U of p and mapping U onto an analytic set A in an open set M of C^N. Let $g=(g_1,\cdots,g_n)$ be an element of $\mathfrak{A}^n$ all of whose level sets on U are countable. Then the set T of all $h=(h_1,\cdots,h_N)$ in $\mathfrak{A}^N$ such that (g,h) is a coordinate set of functions at all points of U has a complement S in $\mathfrak{A}^N$ which is strictly of first category.*

Proof. We may assume that $\bar{U}$ is a compact subset of L. Let $\{W_k\}$ be a sequence of compact subsets of U with $\cup W_k=U$. Let $\tilde{g}_1,\cdots,\tilde{g}_n$ be analytic functions on M with $g_i(p)=\tilde{g}_i(f(p))$ for all p in U and $1\leqq i\leqq n$. For each k let T_k consist of all h in $\mathfrak{A}^N$ such that there exist analytic functions $\bar{h}_1,\cdots,\bar{h}_N$ on M for which $h_i(p)=\bar{h}_i(f(p))$ for all p in U and $1\leqq i\leqq N$ with the property that $(\tilde{g},\bar{h})$ has rank N at each point of $f(W_k)$. Clearly $T\supset\cap T_k$. Thus it is enough to show that for each k the complement S_k of T_k in $\mathfrak{A}^N$ is strictly of first category. If we let J_k be the closure of S_k in $(C(\bar{U}))^N$ it is sufficient to show that for each k

(a) T_k is dense in $\mathfrak{A}^N$

and

(b) For each h in T_k the restriction of h to $\bar{U}$ does not belong to J_k.

We first prove (a). To this end consider h in $\mathfrak{A}^N$. By induction on j, $0\leqq j\leqq N$, we show that there exist functions $h_1',\cdots,h_j'$ in $\mathfrak{A}$ arbitrarily close to $h_1,\cdots,h_j$ respectively and $\bar{h}_1',\cdots,\bar{h}_j'$ as above such that for each r, $0\leqq r\leqq N$, the set of points in A at which the rank of $(\tilde{g},\bar{h}_1',\cdots,\bar{h}_j')$ on M is at most r has dimension at most $\max\{-1,r-j\}$. This is true for $j=0$, for if $\tilde{g}$ had rank r or less on a set $B\subset A$ of dimension $r+1$ then the level sets of g on the regular points of B would not be countable, contrary to the hypothesis on g. Thus we need only show how to find h_{j+1}', assuming that $j<N$ and $h_1',\cdots,h_j'$ have been found. For $0\leqq r\leqq N$ let A_r be those points in A at which the rank of $(\tilde{g},\bar{h}_1',\cdots,\bar{h}_j')$ on M is at most r. By the induction hypothesis either A_r is void or $\dim A_r\leqq r-j$. We consider $K\cup M$ as a topological space in which K and M are disjoint open sets with their given topologies. For $1\leqq i\leqq N$ let the function $\hat{f}_i$ be defined on $K\cup M$ by $\hat{f}_i(p)=f_i(p)$ if $p\in K$ and $\hat{f}_i(z)=z_i$ if $z=(z_1,\cdots,z_N)\in M$.

Similarly define the function $\hat{h}_{j+1}$ on $K \cup M$ by $\hat{h}_{j+1}(p) = h_{j+1}(p)$ for p in K and $\hat{h}_{j+1}(z) = \bar{h}_{j+1}(z)$ for z in M, for a fixed choice of $\bar{h}_{j+1}$. Let $\hat{\mathfrak{A}}$ be the closed subalgebra of $C(K \cup M)$ generated by the functions $\hat{f}_1, \cdots, \hat{f}_N, \hat{h}_{j+1}$, so that $\hat{K}$ is a partially analytic space. If p is any regular point of $A_r - A_{r-1}$ it is easy to see that the set of all $\hat{h}_{j+1}'$ in $\hat{\mathfrak{A}}$ with the property that $(\tilde{g}, \bar{h}_1', \cdots, \bar{h}_j', \hat{h}_{j+1}')$ have rank $r+1$ on M at p form an open dense set in $\hat{\mathfrak{A}}$. Thus there exists $\hat{h}_{j+1}'$ in $\hat{\mathfrak{A}}$ arbitrarily close to $\hat{h}_{j+1}$ such that for each r, $0 \leqq r \leqq N$, the functions $(\tilde{g}, \bar{h}_1', \cdots, \bar{h}_j', \hat{h}_{j+1}')$ have rank $r+1$ at a dense set of regular points of $A_r - A_{r-1}$. Let h_{j+1}' be the restriction of $\hat{h}_{j+1}'$ to K and $\bar{h}_{j+1}'$ the restriction of $\hat{h}_{j+1}'$ to M. It follows that $(\tilde{g}, \bar{h}_1', \cdots, \bar{h}_{j+1}')$ can have rank r or less only on the union of A_{r-1} and a subset of A_r of dimension at most $\max\{-1, r-j-1\}$; this union therefore has dimension at most $\max\{-1, r-j-1\}$. Thus h_{j+1}' has the desired properties. By putting $j = N$ and $r = N-1$ we see that $(\tilde{g}, \bar{h}_1', \cdots, \bar{h}_N')$ have rank N at each point of A, thus proving (a).

To prove (b), consider h in T_k. Thus $\bar{h}$ can be found so that $(\tilde{g}, \bar{h})$ has rank N of all points of $f(W_k)$. Now if h_0 in $\mathfrak{A}^N$ is close enough to h on the set $\bar{U}$ then by a theorem of Grauert and Remmert ([5], p. 291) $\bar{h}_0$ can be made arbitrarily close to $\bar{h}$ on some fixed neighborhood in M of $f(W_k)$. It follows that if h_0 is sufficiently close to h on $\bar{U}$ then $\bar{h}_0$ can be found such that $(\tilde{g}, \bar{h}_0)$ has rank N at all points of $f(W_k)$, so that h_0 is not in S_k. This clearly implies that h is not in J_k, thus proving (b).

The following lemma replaces Lemma 6 of [1]. The proof is the usual proof of this type of lemma.

LEMMA 5. *Let L be an embryonic frame, U an open subset of L, and S a compact subset of U such that for each p in $U - S$ there exists f in $\mathfrak{A}$ with*

$$|f(p)| > \sup\{|f(q)| : q \in S\}.$$

Let $\mathfrak{A}$ separate points of U. Let $1 \in \mathfrak{A}$. Let $\mathfrak{A}_0$ be the uniform closure of $\mathfrak{A}$ in $C(S)$. Then to every non-zero continuous homomorphism σ of $\mathfrak{A}_0$ into the complex numbers corresponds p_0 in S such that $\sigma(f) = f(p_0)$ for all f in $\mathfrak{A}_0$.

Proof. There exists an analytic polyhedron P with frame L, defined by functions $f = (f_1, \cdots, f_n)$ in $\mathfrak{A}$, such that $S \subset P \subset U$. By adding extra functions to f if necessary, we may assume by Theorems 7, 8, and 9 that f is a set of coordinates at each point of P and that f maps P one to-one onto an analytic set $A \subset E^n$. Thus each function in $\mathfrak{A}$ gives rise to an analytic function on A which can be extended to an analytic function on E^n. It follows that if S is identified with the set $f(S) \subset A$ then $\mathfrak{A}_0$ may be identified with the set of all uniform limits on $f(S)$ of functions analytic in E^n. Let z_0

be the point $z_0 = (\sigma(f_1), \cdots, \sigma(f_n))$ of E^n. To prove the lemma it will be sufficient to show that $z_0 \in f(S)$, for then p_0 can be taken to be that point of S with $f(p_0) = z_0$. To this end it is enough to show that for each z in $E^n - f(S)$ there exists h analytic on E^n with $|h(z)| > \sup\{|h(w)| : w \in f(S)\}$. This is clear if $z \in E^n - A$ for then there exists h which vanishes on A but not at z. If $z \in A - f(S)$ then by the hypothesis of the lemma there exists g in $\mathfrak{A}$ with $|g(p)| > \sup\{|g(q)| : q \in S\}$. If we let h be any function on E^n with $h \circ f = g$ on P, then h satisfies the requirement.

Corollary. *Under the hypotheses of Lemma 5 if S_1 and S_2 are disjoint closed sets with $S = S_1 \cup S_2$ then the function equal to 1 on S_1 and 0 on S_2 is in $\mathfrak{A}_0$.*

The next theorem extends Theorem 8 of [1]. The proof is the same.

Theorem 10. *Let K be a holomorphically convex analytic space such that $1 \in \mathfrak{A}$ and $\mathfrak{A}$ separtes point of K. Let $f = (f_1, \cdots, f_n)$ be an almost proper map of K into C^n, $f_i \in \mathfrak{A}$. Let $\{S_k\}$ be a sequence of compact subsets of K such that $\cup S_k = K$ and $S_k \subset \operatorname{int} S_{k+1}$ for all k. Let*

$$J_k = \{z \in C^n : |z_i| \leqq k, 1 \leqq i \leqq n\}.$$

Let H_k be any compact subset of $f^{-1}(J_k)$ which is open in $f^{-1}(J_k)$ and contains $(S_k \cap f^{-1}(J_k)) \cup H_{k-1}$. Write

$$G_k = [f^{-1}(J_k) \cap H_{k+1}] - H_k$$

for $k \geqq 1$ and $G_0 = H_1$. For $k = 0, 1, \cdots$ let λ_k be any function in $\mathfrak{A}$ and ϵ_k any positive number. Let Γ be the set of all $g = (g_1, \cdots, g_N)$ in $\mathfrak{A}^N$ such that for all k, $k = 0, 1, \cdots$

$$|g_1(p) - \lambda_k(p)| < \epsilon_k, \quad p \text{ in } G_k.$$

Then Γ is not strictly of first category in $\mathfrak{A}^N$. If the sequence $\{\epsilon_k\}$ is bounded and if

$$\inf\{|\lambda_k(p)| : p \in G_k\} \to \infty \quad as \ k \to \infty$$

then (f, g_1) is a proper map of K into C^{n+1}.

Proof. Let $\{D_k\}$ be a sequence of compact subsets of K and F_k a closed subset of $(C(D_k))^N$ such that for each k the set $\tilde{F}_k$ of all g in $\mathfrak{A}^N$ whose restriction to D_k is in F_k is nowhere dense in $\mathfrak{A}^N$. To show that Γ is not strictly of first category we must find g in $\Gamma - \cup \tilde{F}_k$. By re-ordering and padding the sequence $\{D_k\}$ if necessary we may assume that $D_k \subset H_k$ for all k. We also assume that $\{\epsilon_k\}$ is a monotonely decreasing sequence. If $z = (z_1, \cdots, z_N)$ is an N-tuple of complex numbers define $|z| = \max\{|z_i|\}$. We construct by induction a sequence $g^0, g^1, \cdots$ of elements in $\mathfrak{A}^N$ such that

(a) $g^0 + \cdots + g^k \notin \bar{F}_k$ for $k \geqq 1$,

(b) $|g^k(p)| < 2^{-k-1} \min\{\epsilon_k, \delta_1, \cdots, \delta_{k-1}\}$ for $k \geqq 1$ and p in H_k, where δ_i is the distance in $(C(D_i))^N$ of $g^0 + \cdots + g^i$ to F_i,

and

(c) $|g^0{}_1(p) + \cdots + g^k{}_1(p) - \lambda_k(p)| < \frac{1}{2}\epsilon_k$, $p \in G_k$ and $k = 0, 1, \cdots$.

To begin the induction take $g^0{}_1 = \lambda_0, g^0{}_2, \cdots, g^0{}_N$ arbitrary. Assume that $g^0, \cdots, g^{k-1}$ have been constructed to satisfy the relevant conditions. Let U be a neighborhood of H_{k+1} such that $H_{k+1} = U \cap f^{-1}(J_{k+1})$. Thus for each p in $U - H_{k+1}$ for some i with $1 \leqq i \leqq n$ we have

$$|f_i(p)| > k + 1 \geqq \sup\{|f_i(q)| : q \in H_{k+1}\}.$$

Since $G_k \cup H_k = f^{-1}(J_k) \cap H_{k+1}$ we see for p in

$$H_{k+1} - (G_k \cup H_k) \subset f^{-1}(J_{k+1}) - f^{-1}(J_k)$$

that

$$|f_i(p)| \geqq k + 1 > k \geqq \sup\{|f_i(q)| : q \in G_k \cup H_k\}$$

for some i. It follows from Lemma 5 that there exists τ in $\mathfrak{A}$ arbitrarily near to 0 on H_k and to 1 on G_k. Write

$$g^k{}_1 = -\tau(g^0{}_1 + \cdots + g^{k-1}{}_1 - \lambda_k)$$

and $g^k{}_i = 0$, $2 \leqq i \leqq N$. It is clear that if τ is chosen correctly then g^k satisfies (b) and (c). Thus by changing g^k slightly we may also assume that g^k satisfies (a).

Define $g = \sum_{k=0}^{\infty} g^k$. For each k this series converges uniformly on H_k because $|g^j(p)| < \epsilon_j 2^{-j-1}$ for $p \in H_k$ whenever $j > k$. Thus $g \in \mathfrak{A}$, since for any compact $S \subset K$ there exists j with $S \subset S_j$, so that $S \subset H_k$ if k is chosen so large that $f(S_j) \subset J_k$. We see by (b) and (c) that for all p in G_k

$$|g_1(p) - \lambda_k(p)| \leqq |g^0{}_1(p) + \cdots + g^k{}_1(p) - \lambda_k(p)| + \sum_{j=k+1}^{\infty} |g^j{}_1(p)| \leqq \tfrac{1}{2}\epsilon_k + \sum_{j=k+1}^{\infty} \epsilon_j 2^{-j-1} < \epsilon_k.$$

Therefore $g \in \Gamma$. Also the distance in $(C(D_k))^N$ of g to F_k is at least

$$\delta_k - \sum_{j=k+1}^{\infty} \sup\{|g^j(p)| : p \in D_k \subset H_k\} \geqq \delta_k - \sum_{j=k+1}^{\infty} 2^{-j-1}\delta_k > 0.$$

This completes the proof that Γ is not strictly of first category. The rest of the theorem is proved as in [1].

We turn now to a consideration of certain imbedding properties. By Corollary 1 of Theorem 8 we see that every holomorphically convex embryonic space with a separating algebra $\mathfrak{A}$ of analytic functions can be realized as an

analytic space which is holomorphically convex relative to the given algebra $\mathfrak{A}$, and by Theorem 6 in turn we see that the analytic space can be realized as a holomorphically convex embryonic space with a separating algebra $\mathfrak{A}$ of analytic functions. For this reason Theorem 10 holds for arbitrary holomirpically convex analytic or embryonic spaces, and the following theorem also yields information for such spaces.

THEOREM 11. *Let K be a holomorphically convex analytic space such that $1 \in \mathfrak{A}$ and $\mathfrak{A}$ separates points of K. Let K have dimension at most n at each point, so that by Theorem 2 there exists an almost proper map $f = (f_1, \cdots, f_n) \in \mathfrak{A}^n$ of K into C^n. Let $L_1, \cdots, L_k$ be a finite family of frames in K, and $U_1, \cdots, U_k$ open subsets of these respective frames such that there exist integers $N_1, \cdots, N_k$ such that each point p of U_i, $1 \leqq i \leqq k$, is an analytic point of L_i of rank at most N_i. Write*

$$N = \max\{n+1, N_1, \cdots, N_k\}.$$

Then there exists $g = (g_1, \cdots, g_N)$ in $\mathfrak{A}^N$ such that

(a) *$(f, g_1, \cdots, g_{n+1})$ separates points of K,*
(b) *For each i, $1 \leqq i \leqq k$, the functions $(f, g_1, \cdots, g_{N_i})$ are coordinates at each point of U_i,*
(c) *(f, g_1) is a proper map of K into C^{n+1}.*

Proof. By Lemma 2 the level sets of f on K are all countable. By Theorem 7 the set T_1 of all g for which (a) is not true is strictly of first category in $\mathfrak{A}^N$. By Theorem 9 the set T_2 of all g for which (b) is not true is strictly of first category in $\mathfrak{A}^N$. Thus $T_1 \cup T_2$ is strictly of first category. By Theorem 10 there exists g in $K - (T_1 \cup T_2)$ which has property (c). This completes the proof.

Now we would like to improve Theorem 10 by choosing g so that the mapping $(f_1, \cdots, f_\gamma, g_1)$ is proper on certain embryonic frames L in K, where $\gamma = \dim L$ and $(f_1, \cdots, f_\gamma)$ is almost proper on L. Instead of showing that g can be so chosen we show in the next theorem that the map f can be chosen so that g_1 automatically satisfies the extra conditions. This theorem will at the same time be a considerable strengthening of Theorem 2.

THEOREM 12. *Let $\{L_j\}$ be a finite family of analytic sets of finite dimensions $\{\gamma = \gamma(j)\}$ in a d-dimensional holomorphically convex analytic space K. Then if $\mathfrak{A}$ is separating there exists $f = (f_1, \cdots, f_d)$ in $\mathfrak{A}^d$ such that*

(a) *For each j the mapping $f^j = (f_1, \cdots, f_\gamma)$ of L_j into C^γ is almost proper,*
(b) *For each j the functions $f_{\gamma+1}, \cdots, f_d$ vanish on L_j.*

Sketch of Proof. By decomposing each L_j into its homogeneous-dimensional parts we may assume that each L_j is a special frame. By lumping together all L_j of a given dimension we may assume that the L_j are special frames $L_1, L_2, \cdots, L_d$ of dimensions $1, 2, \cdots, d$ respectively. Finally, by enlarging the L_j appropriately we assume $L_1 \subset L_2 \subset \cdots \subset L_d = K$.

Now let S be any compact subset of K for which $S = \tilde{S}$. We show by induction on k, $0 \leqq k \leqq d$, that there exist $h_1, \cdots, h_k$ in $\mathfrak{A}$ each of absolute value less than 1 on S such that for $1 \leqq j \leqq k$ the functions $h_{j+1}, \cdots, h_k$ vanish on L_j and such that $h_1, \cdots, h_j$ define an analytic polyhedron P_j in L_j with $S \cap L_j \subset P_j$. Since the case $k = 0$ is trivial, we may assume that $h_1, \cdots, h_{k-1}$ have been found with the desired properties. Consider p in $K - L_{k-1} - S$. Thus there exists w_1 in $\mathfrak{A}$ vanishing on L_{k-1} with $w_1(p) = 1$. Since $S = \tilde{S}$ there exists w_2 in $\mathfrak{A}$ with $w_2(p) = 1$ and $|w_1(q) w_2(q)| < 1$ for all q in S. Thus there exist finitely many functions $g_1, \cdots, g_N$ in $\mathfrak{A}$ which vanish on L_{k-1} and have absolute values less than 1 on S such that $h_1, \cdots, h_{k-1}$, $g_1, \cdots, g_N$ define an analytic polyhedron Q in L_k with $S \cap L_k \subset Q$. By the construction used to prove Theorem 3 of [1], we may successively reduce the number N of such functions g, obtaining at the last step functions $h_1', \cdots, h_{k-1}'$, g_1' in $\mathfrak{A}$ having the desired properties for the functions $h_1, \cdots, h_k$ at the k-th stage. Thus we may assume that the functions $h_1, \cdots, h_d$ can be chosen for each compact set S with $S = \tilde{S}$. Once this is done the proof of Theorem 4 of [1] can be repeated word for word to construct the desired functions $f_1, \cdots, f_d$.

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ARITHMETICAL NOTES, V. A DIVISIBILITY PROPERTY OF THE DIVISOR FUNCTION.*

By Eckford Cohen.

1. **Introduction.** Let k and n denote positive integers, k fixed, and let $\tau(n)$ denote the number of (positive) divisors of n. The set of all n such that $k \mid \tau(n)$ will be denoted S_k and, for real $x \geqq 1$, the enumerative function (§ 2) of S_k will be denoted $S_k(x)$. Sathe [5] has investigated $S_k(x)$ for odd $k > 1$, obtaining the estimate,

$$S_k(x) \sim \alpha_k x \quad \text{as} \quad x \to \infty, \tag{1.1}$$

where α_k is a positive constant defined by (3.6). Sathe's proof depends upon a result [5, (3)] of S. Selberg and Pillai which is substantially equivalent to the Prime Number Theorem. It is the purpose of this note to prove a refinement of Sathe's estimate (Theorem 3.1). The method of the paper is elementary; in particular, no appeal is made to the theory of primes.

2. **Preliminaries.** The characteristic function $s(n)$ of a set S of positive integers n is defined by $s(n) = 1$ or 0 according as $n \in S$ or $n \notin S$. The enumerative function $S(x)$ of S is the summatory function of $s(n)$, that is, $S(x) = \sum_{n \leqq x} s(n)$. In this note Q will denote the set of square-free integers, and the characteristic and enumerative functions of Q will be denoted $q(n)$ and $Q(x)$, respectively. The symbol $a_k(n)$ will denote the characteristic function of the set A_k of the k-th powers. The Dedekind ψ-function is the arithmetical function,

$$\psi(n) = \sum_{d\delta = n} q(d)\delta, \qquad \psi(n)/n = \prod_{p \mid n} (1 + 1/p), \tag{2.1}$$

where the product is over the prime divisors of n. The number of square-free divisors of n will be denoted $\theta(n)$.

We recall a previously proved lemma.

Lemma 2.1 ([2, Lemma 5.2]). *For positive integral r,*

$$Q_r^*(x) \equiv \sum_{\substack{n \leqq x \\ (n,r)=1}} q(n) = \beta(r)x + O(x^{\frac{1}{2}}\theta(r)) \tag{2.2}$$

where $\beta(r) = (6/\pi^2)r/\psi(r)$.

* Received May 24, 1961.

Remark 2.1. Clearly $\beta(n)$ is bounded.

Next we note a trivial though fundamental principle.

Remark 2.2. An integer n is *uniquely* representable as the product of a k-free integer by a k-th power.

Let $t_{k,r}(n)$ denote the characteristic function of the set $T_{k,r}$ of integers with a k-th power divisor whose complementary divisor is square-free and prime to r. Then by Remark 2.2

$$t_{k,r}(n) = \sum_{\substack{d\delta=n \\ (\delta,r)=1}} a_k(d)q(\delta). \tag{2.3}$$

We deduce now an estimate for the enumerative function $T_{k,r}(x)$ of the sequence $T_{k,r}$. Let $\zeta(s)$ denote the Riemann ζ-function, $s > 1$.

LEMMA 2.2. *For positive integral r and fixed $k \geqq 3$*

$$T_{k,r}(x) \equiv \sum_{n\leqq x} t_{k,r}(n) = \beta(r)\zeta(k)x + O(x^{1/k}\beta(r)) + O(x^{\frac{1}{2}}\theta(r)). \tag{2.4}$$

Proof. By (2.3) and Lemma 2.1,

$$\begin{aligned} T_{k,r}(x) &= \sum_{\substack{d\delta\leqq x \\ (\delta,r)=1}} a_k(d)q(\delta) = \sum_{n\leqq x} a_k(n)Q_r{}^*(x/n) \\ &= \beta(r)x\sum_{n\leqq x} a_k(n)n^{-1} + O(x^{\frac{1}{2}}\theta(r)\sum_{n\leqq x} a_k(n)n^{-\frac{1}{2}}) \\ &= \beta(r)x\sum_{n=1}^{\infty} n^{-k} + O(\beta(r)x\sum_{n>x^{1/k}} n^{-k}) + O(x^{\frac{1}{2}}\theta(r)\sum_{n\leqq x^{1/k}} n^{-k/2}), \end{aligned}$$

from which (2.4) results.

Let the integer n have the canonical factorization

$$n = p_1^{e_1}\cdots p_t^{e_t}, \qquad t=0 \text{ if } n=1, \tag{2.5}$$

where $p_1,\cdots,p_t$ denote the distinct prime factors of n. The set B of integers n in (2.5) with $e_i > 1$ for all i is the set of "square-full" integers. Let $B(x)$ represent the enumerative function of B and $b(n)$ its characteristic function.

LEMMA 2.3.

$$B(x) \equiv \sum_{n\leqq x} b(n) = O(x^{\frac{1}{2}}). \tag{2.6}$$

Proof. This estimate results from a much sharper approximation for $B(x)$ proved by Bateman in [1] using an elementary approach.

3. The main result. We introduce some further notation. The set of

k-free integers will be denoted Q_k, while the set of k-full integers namely, the integers n in (2.5) with $e_i \geqq k$ $(1 \leqq i \leqq t)$, will be represented by B_k. In particular, $Q = Q_2$ and $B = B_2$. If r is a positive integer $\leqq k$, then $F_{k,r}$ will be used to denote the intersection, $B_r \cap Q_k$, and $G_{k,r}$ the intersection, $F_{k,r} \cap S_k$. For simplicity, we write $F_k = F_{k,2}$, $G_k = G_{k,2}$, and use $g_k(n)$ and $G_k(x)$ to denote, respectively, the characteristic and enumerative functions of G_k.

Remark 3.1. It will be observed that $g_k(n) = 1$ if n is of the form (2.5) with $2 \leqq e_i \leqq k-1$ for all i and if $k \mid \tau(n)$, otherwise $g_k(n) = 0 (k \geqq 2)$. In particular, $G_k \subseteq F_k$.

The following representation for $s_k(n)$, the characteristic function of S_k, is basic for the main theorem.

LEMMA 3.1. *If k is odd and $\geqq 3$, then*

$$s_k(n) = \sum_{d\delta = n} g_k(d) t_{k,d}(\delta). \tag{3.1}$$

Proof. It may be seen (Remark 2.2) that an integer n has a *unique* factorization of the form,

$$n = d_1 d_2 d_3, \quad (d_1, d_3) = 1, \quad d_1 \in F_k, \quad d_2 \in A_k, \quad d_3 \in Q. \tag{3.2}$$

Moreover, if n is defined by (2.5), then $\tau(n) = (e_1 + 1) \cdots (e_t + 1)$, $\tau(1) = 1$. Hence, in the factorization (3.2), since k is odd and > 1,

$$n \in S_k \Longleftrightarrow d_1 \in G_k.$$

Thus, one obtains

$$s_k(n) = \sum_{\substack{d_1 d_2 d_3 = n \\ (d_1, d_3) = 1}} g_k(d_1) a_k(d_2) q(d_3),$$

which by (2.3) is the same as (3.1).

We mention two simple results that will be required. (Convention: vacuous products will be assumed to have the value 1).

LEMMA 3.2. *If $\epsilon < 1/r(r-1)$, then*

$$\sum_{\substack{n=1 \\ n \in F_{k,r}}} n^{\epsilon - 1/(r-1)} \qquad (k \geqq r \geqq 2) \tag{3.3}$$

is convergent.

Proof. We have

$$\sum_{\substack{n \leqq x \\ n \in F_{k,r}}} n^{\epsilon - 1/(r-1)} \leqq \prod_{i=r}^{k-1} \Big(\sum_{\substack{n \leqq x \\ n \in A_i}} n^{\epsilon - 1/(r-1)} \Big) \leqq \prod_{i=r}^{k-1} \zeta(i(1/(r-1) - \epsilon)).$$

Thus the convergence of (3.3) is assured.

LEMMA 3.3. *If $x \geqq 2$, then*

$$\sum_{n \leqq x} \tau(n)/n = O(\log^2 x). \tag{3.4}$$

For a much sharper (elementary) estimate of the sum on the left of (3.4), see [4, Lemma 71.3].

We are now in a position to prove our main result.

THEOREM 3.1. *If k is a fixed odd integer $\geqq 3$, then for $x \geqq 2$*

$$S_k(x) = \alpha_k x + O(x^{\frac{1}{2}} \log^2 x), \tag{3.5}$$

where

$$\alpha_k = \zeta(k) \sum_{n=1}^{\infty} g_k(n)\beta(n)n^{-1} \equiv 6\pi^{-2}\zeta(k) \sum_{n=1}^{\infty} g_k(n)/\psi(n). \tag{3.6}$$

Proof. We first remark that the series in (3.6) is absolutely convergent, by the boundedness of $\beta(n)$ and the case $r = 2$ of Lemma 3.2 with $\epsilon = 0$.

By Lemma 3.1, it follows that

$$S_k(x) \equiv \sum_{n \leqq x} s_k(n) = \sum_{d\delta \leqq x} g_k(d) t_{k,d}(\delta) = \sum_{n \leqq x} g_k(n) T_{k,n}(x/n),$$

and hence, by Lemma 2.2 and Remark 2.1, that

$$\begin{aligned} S_k(x) = \zeta(k) x \sum_{n \leqq x} g_k(n)\beta(n)n^{-1} &+ O(x^{1/k} \sum_{n \leqq x} g_k(n) n^{-1/k}) \\ &+ O(x^{\frac{1}{2}} \sum_{n \leqq x} g_k(n)\theta(n)n^{-\frac{1}{2}}). \end{aligned} \tag{3.7}$$

Noting again that $\beta(n) = O(1)$, one obtains

$$\sum_{n \leqq x} g_k(n)\beta(n)n^{-1} = \sum_{n=1}^{\infty} g_k(n)\beta(n)n^{-1} + O(\sum_{n > x} g_k(n)n^{-1}).$$

Appealing to Lemma 3.2 ($r = 3$), it follows (for some $C_k > 0$ depending on k alone) that

$$\begin{aligned} \sum_{n > x} g_k(n)n^{-1} \leqq \sum_{\substack{n^2 m > x \\ m \in F_{k,3}}} 1/n^2 m = \sum_{\substack{m=1 \\ m \in F_{k,3}}}^{\infty} 1/m \sum_{n > (x/m)^{\frac{1}{2}}} 1/n^2 \\ \leqq C_k x^{-\frac{1}{2}} \sum_{\substack{m=1 \\ m \in F_{k,3}}}^{\infty} m^{-\frac{1}{2}} = O(x^{-\frac{1}{2}}). \end{aligned}$$

Therefore

$$\sum_{n \leqq x} g_k(n)\beta(n)n^{-1} = \zeta^{-1}(k)\alpha_k + O(x^{-\frac{1}{2}}). \tag{3.8}$$

Noting that $G_k \subset B$, one obtains by partial summation and (2.6)

$$\sum_{n\leqq x} g_k(n)n^{-1/k} = \sum_{n\leqq x} G_k(n)\cdot n^{-1/k}(1-(1+1/n)^{-1/k}) + O(G_k(x)x^{-1/k})$$

$$= O(\sum_{n\leqq x} B(n)n^{-(1+1/k)} + O(B(x)x^{-1/k})$$

$$= O(\sum_{n\leqq x} n^{-(1/k+\frac{1}{2})}) + O(x^{\frac{1}{2}-1/k}),$$

from which it results that

$$\sum_{n\leqq x} g_k(n)n^{-1/k} = O(x^{\frac{1}{2}-1/k}). \tag{3.9}$$

To complete the proof, we note the following simple properties of $\theta(n)$: $\theta(n)$ is multiplicative, $\theta(n^2) = \theta(n)$, and $\theta(n) = O(\tau(n)) = O(n^\epsilon)$ for all $\epsilon > 0$. Hence by Lemma 3.2 with $r = 3$, $0 < \epsilon < 1/6$, one deduces that

$$\sum_{n\leqq x} g_k(n)\theta(n)n^{-\frac{1}{2}} \leqq \sum_{\substack{n^2m\leqq x\\(n,m)=1\\m\in F_{k,3}}} \theta(n^2m)(n^2m)^{-\frac{1}{2}} = \sum_{n\leqq x} \theta(n^2)/n \sum_{\substack{m\leqq x\\m\in F_{k,3}}} \theta(m)m^{-\frac{1}{2}}$$

$$= O(\sum_{n\leqq x} \theta(n)/n) = O(\sum_{n\leqq x} \tau(n)/n);$$

therefore by Lemma 3.3,

$$\sum_{n\leqq x} g_k(n)\theta(n)n^{-\frac{1}{2}} = O(\log^2 x). \tag{3.10}$$

The theorem follows on combining (3.7), (3.8), (3.9), and (3.10).

Remark 3.2. For the case in which k is a prime, a direct evaluation of α_k has been given by Sathe [5, Theorem 6]. For a much simpler, indirect evaluation of α_k in this case, the reader is referred to [3].

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ON BOUNDARY VALUE PROBLEMS FOR LAPLACE'S EQUATION.*

By R. T. Seeley.[1]

Introduction. The object of this paper is to formulate certain boundary value problems for Laplace's equation in a way suitable for the representation of the boundary condition by a singular integral equation. This formulation is modeled to some extent on the Abel summation of Fourier series. The application considered is to show that, for a first order boundary operator that "covers" the Laplacian, the index of the problem is zero.

Section 1 describes the regions considered, and develops the results used to justify the representation of the boundary conditions by singular integral equations. Some lemmas used in this are proved in Section 3. Section 2 discusses the question of the index.

The methods used here may prove applicable to wide classes of elliptic boundary problems.

1. On the connection between u and $\partial u/\partial n$ for a harmonic function u. Let $k \geqq 2$, and G be a bounded region in $k+1$ dimensional Euclidean space E^{k+1}, with boundary Γ. We assume that Γ is connected, and is almost a C_2 manifold, of dimension k. Specifically

i) Γ is a C_1 manifold; the unit normal at the point x in Γ is denoted by $\boldsymbol{\nu}(x)$;

ii) there is a number $\tau > 0$ such that for every x in Γ the sphere of radius τ about $x \pm \tau\boldsymbol{\nu}(x)$ intersects Γ only at x;

iii) for each t in $0 \leqq |t| \leqq \tau$, the points $\{x + t\boldsymbol{\nu}(x);\ x \text{ in } \Gamma\}$ are all distinct and form a C_1 manifold Γ_t, bounding the region G_t; and

iv) if we introduce in E^{k+1} orthonormal coordinates $\xi_1, \cdots, \xi_k, \zeta$ with origin at the point x in Γ in such a way that the tangent plane to Γ at x is $\zeta = 0$, then in a neighborhood of x, Γ can be represented by $\zeta = h(\xi_1, \cdots, \xi_k)$, with h a C_1 function.

These conditions are not independent, of course.

It follows from these conditions that Γ_t is of the same type as Γ, with τ

* Received July 13, 1961.

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replaced by $\tau - |t|$. Since the point $\boldsymbol{x}'$ on Γ_t nearest to the point $\boldsymbol{x}$ on Γ is $\boldsymbol{x}' = \boldsymbol{x} + t\boldsymbol{\nu}(\boldsymbol{x})$, $\boldsymbol{\nu}(\boldsymbol{x})$ is the normal to Γ_t at $\boldsymbol{x} + t\boldsymbol{\nu}(\boldsymbol{x})$.

It will be convenient to write simply $\boldsymbol{\nu}$ instead of $\boldsymbol{\nu}(\boldsymbol{x})$ in most contexts.

Let $d\sigma$ be the natural surface measure on Γ, and $|\Gamma| = \int_\Gamma d\sigma$. For a function f in $L^1(\Gamma)$, and $\boldsymbol{x}$ in E^{k+1}, let $Pf(\boldsymbol{x}) = c\int_\Gamma f(\boldsymbol{y})\,|\boldsymbol{x} - \boldsymbol{y}|^{1-k} d\sigma_y$, where $c = \frac{1}{2}\Gamma((k-1)/2)\Pi^{-(k+1)/2}$, and $|\boldsymbol{x} - \boldsymbol{y}|^2 = \sum_1^{k+1}(x_i - y_i)^2$. $Pf(\boldsymbol{x})$ is harmonic when $\boldsymbol{x}$ is not on Γ; and since locally on Γ, P is dominated by convolution (in local coordinates) with a function in L^1, $Pf(\boldsymbol{x})$ is defined almost everywhere on Γ. For each t in $0 < |t| \leqq \tau$ and $\boldsymbol{x}$ in Γ we define $P_tf(\boldsymbol{x})$ by $P_tf(\boldsymbol{x}) = Pf(\boldsymbol{x} + t\boldsymbol{\nu})$. Lemma 1 proves that $P_tf \to P_0f$ in $L^1(\Gamma)$, and in $L^p(\Gamma)$ if f is in $L^p(\Gamma)$ $(1 \leqq p \leqq \infty)$; here P_0f is the restriction to Γ of Pf.

An operator N, the normal derivative, can be defined on the range of P_0 by $NP_0f(\boldsymbol{x}) = \lim_{t\to 0-} D_tP_tf(\boldsymbol{x})$, where D_t denotes differentiation with respect to t. Lemma 2 shows that there is an operator K_0 (whose kernel is the adjoint of that for the double layer potential) such that $D_tP_tf \to f + K_0f$ in $L^p(\Gamma)$ as $t \to 0-$; the convergence is uniform if f is continuous. Thus $NP_0 = I + K_0$. The lemma shows moreover that if f is bounded, then K_0f is continuous; and that the kernel of K_0 is uniformly $0(|\boldsymbol{x} - \boldsymbol{y}|^{1-k})$.

The operator P_0 on $L^1(\Gamma)$ has trivial null space, as the following argument shows. Since $NP_0 = I + K$, the null space of P_0 is contained in that of $I + K_0$. This latter null space contains only continuous functions; for we can choose n so that the kernel of $K_0{}^n$ is bounded, so that if $f = -K_0f$, then $f = (-1)^n K_0{}^n f$ is bounded, from which it follows by Lemma 2 that $f = -K_0f$ is continuous. Thus if $P_0f = 0$ for f in $L^1(\Gamma)$, then f is continuous, and by Lemma 1 P_0f is the boundary value in the classical sense of the function Pf in E^{k+1}. It follows from the maximum principle that $Pf(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$ in E^{k+1}. Now if $\phi(\boldsymbol{x})$ is any function in C_2 on E^{k+1}, with compact support, we have

$$\int_\Gamma \phi(\boldsymbol{x})f(\boldsymbol{x})\,d\sigma_x = c'\int_\Gamma\int_{E^{k+1}} |\boldsymbol{x} - \boldsymbol{y}|^{1-k}\Delta\phi(\boldsymbol{y})f(\boldsymbol{x})\,d\boldsymbol{y}\,d\sigma_x$$

$$= c''\int_{E^{k+1}} \Delta\phi(\boldsymbol{y})Pf(\boldsymbol{y})\,d\boldsymbol{y} = 0.$$

It follows that $f(\boldsymbol{x}) \equiv 0$ on Γ.

We can also show that $NP_0f = 0$ only if P_0f is constant. For we have

seen that if $NP_0f = 0$, then f is continuous; thus Lemmas 1 and 2 yield for $u(\mathbf{x}) = Pf(\mathbf{x})$ that

$$0 = \int_\Gamma u\,\partial u/\partial n = \lim_{t\to 0-} \int_{\Gamma_t} u\,\partial u/\partial n = \lim_{t\to 0-} \int_{G_t} |\nabla u|^2 = \int_G |\nabla u|^2.$$

This shows that $u = Pf$ is constant inside G, so that P_0f is constant.

For f in $L^p(\Gamma)$ and g in $L^q(\Gamma)$ we have $(NP_0f, P_0g) = (P_0f, NP_0g)$; this can be verified by approximating f and g with continuous functions and applying the formula

$$\int_\Gamma \overline{(P_0g)}\,\partial Pf/\partial n = \int_G \overline{(\nabla Pg)}\cdot(\nabla Pf) = \int_\Gamma P_0 f\,\overline{(\partial Pg/\partial n)},$$

or by using the same formula on Γ_t and letting $t \to 0-$. Part of the same argument, with $f = g$, shows that $(NP_0f, P_0f) \geqq 0$.

If A is the averaging operator, i.e. $Af(\mathbf{x}) = |\Gamma|^{-1}\int_\Gamma f(\mathbf{y})\,d\sigma_y$, then $ANP_0 = 0$; for by Lemma 2

$$\int_\Gamma NP_0f = \lim_{t\to 0-}\int_{\Gamma_t} D_tPf(\mathbf{x} + t\boldsymbol{\nu})\,d\sigma_{x'} \qquad (\mathbf{x}' = \mathbf{x} + t\boldsymbol{\nu}),$$

and the latter integral is $\int_{\Gamma_t} \partial u/\partial n$ for a function u harmonic in Γ_t and its interior. Now $(A + N)P_0$ has trivial null space. For if f is in $L^1(\Gamma)$ and $(A + N)P_0f = 0$, then $0 = (A^2 + AN)P_0f = AP_0f$, which gives also $NP_0f = 0$. This last equality implies P_0f is constant, and $0 = AP_0f$ shows that this constant is zero. Thus $(A + N)P_0 = I + K_0 + AP_0 = I + L$, where the kernel $L(\mathbf{x}, \mathbf{y})$ of L differs from the kernel of K_0 by a continuous function of $\mathbf{y}$; and $I + L$ has trivial null space.

Now we can construct a right inverse J_0 for $A + N$, using P_0 as parametrix. Set $J_0 = P_0 + P_0F$, where F is an integral operator to be determined. $(A + N)J_0 = I + L + F + LF$, so we seek F as a solution of $(I + L)F = -L$. F can be determined as an integral operator with kernel $F(\mathbf{x}, \mathbf{y})$ given by $F(\mathbf{x}, \mathbf{y}) + \int_\Gamma L(\mathbf{x}, \mathbf{z})F(\mathbf{z}, \mathbf{y})\,d\sigma_z = -L(\mathbf{x}, \mathbf{y})$. Since $I + L$ has trivial null space when acting on $L^1(\Gamma)$, this equation has, for each $\mathbf{y}$ in Γ, a unique solution $F(\mathbf{x}, \mathbf{y})$ in $L^1(\Gamma)$. The kernel $F(\mathbf{x}, \mathbf{y})$ constructed in this way has a singularity like that of $L(\mathbf{x}, \mathbf{y})$, and so defines an operator F on $L^1(\Gamma)$ such that $F + LF = -L$, as desired.

From what we have shown about P_0, any function P_0f in its range is the boundary value of the harmonic function $u = Pf$ in G, with $\partial u/\partial n = NP_0f$ on Γ, both boundary values being assumed in the L^p sense if f is in $L^p(\Gamma)$.

Thus any function $J_0 f$ in the range of $J_0 = P_0(I+F)$ is the boundary value of a harmonic function $u = P(I+F)f$, and on Γ $\partial u/\partial n = NP_0(I+F)f = NJ_0 f = f - AJ_0 f$. Since $ANP_0 = 0$, we find $Af - A^2 J_0 f$ or, since $A^2 = A$,

1) $$AJ_0 f = Af.$$

Thus $\partial u/\partial n = f - Af$. The function

$$J(\boldsymbol{x},\boldsymbol{y}) = c\,|\,\boldsymbol{x}-\boldsymbol{y}\,|^{1-k} + c\int_\Gamma |\,\boldsymbol{x}-\boldsymbol{z}\,|^{1-k} F(\boldsymbol{z},\boldsymbol{y})\,d\sigma_z$$

with $\boldsymbol{x}$ in E^{k+1} and $\boldsymbol{y}$ in Γ, is then the Neumann function for G.

The results we have just obtained can be formulated as follows.

THEOREM 1. *If f is in $L^p(\Gamma)$, $1 \leqq p < \infty$, then there is a function $u(\boldsymbol{x})$ harmonic in G, given by $u = P(I+F)f$, such that*

$$\int_\Gamma |\, D_t u(\boldsymbol{x} + t\boldsymbol{\nu}) - f(\boldsymbol{x}) + Af\,|^p\, d\sigma \to 0,$$

and

$$\int_\Gamma |\, u(\boldsymbol{x} + t\boldsymbol{\nu}) - J_0 f(\boldsymbol{x})\,|^p d\sigma \to 0,$$

as $t \to 0-$. If f is bounded, $u(\boldsymbol{x} + t\boldsymbol{\nu}) \to J_0 f(\boldsymbol{x})$ uniformly. If f is continuous, $D_t u(\boldsymbol{x} + t\boldsymbol{\nu}) \to f(\boldsymbol{x}) - Af$ uniformly.

In order to state our next result, the uniqueness result corresponding to Theorem 1, in the best way, we need the formula

1a) $$J_0 Af = Af,$$

or what amounts to the same thing, that $J_0\phi = \phi$ if ϕ is constant. Suppose ϕ constant, and let $u = P(I+F)\phi$; then by Theorem 1 $\partial u/\partial n = \phi - A\phi = 0$. This boundary value is assumed uniformly, so by classical potential theory u is constant. Hence on Γ, $u = J_0\phi$ is constant. To evaluate the constant, we take averages and use (1) to obtain $J_0\phi = AJ_0\phi = A\phi = \phi$, which establishes 1a).

We can now establish the following converse of Theorem 1.

THEOREM 2. *If u is harmonic in G, and f is a function in $L^p(\Gamma)$, $(1 \leqq p < \infty)$, such that $\int_\Gamma |\, D_t u(\boldsymbol{x} + t\boldsymbol{\nu}) - f(\boldsymbol{x})\,|^p d\sigma \to 0$ as $t \to 0-$, then there is a function g in $L^p(\Gamma)$ such that $\int_\Gamma |\, u(\boldsymbol{x} + t\boldsymbol{\nu}) - g(\boldsymbol{x})\,|^p d\sigma \to 0$ as $t \to 0-$, and $g = J_0(f + Ag)$.*

The theorem is easy to establish if we assume that $\partial u/\partial n$ and u assume uniformly their respective boundary values f and g. For if $D_t u(\boldsymbol{x}+t\boldsymbol{\nu})\to f(\boldsymbol{x})$ uniformly as $t\to 0-$, and for some g $u(\boldsymbol{x}+t\boldsymbol{\nu})\to g(\boldsymbol{x})$ uniformly as $t\to 0-$, then the function $v(\boldsymbol{x})=u(\boldsymbol{x})-P(I+F)f(\boldsymbol{x})$ satisfies $\partial v/\partial n=0$ on Γ, in the classical sense, so v is constant. Since f is $\partial u/\partial n$, we have $\int_\Gamma f=0$, by (1), $\int_\Gamma P(I+F)f=\int_\Gamma J_0 f=\int_\Gamma f=0$. Then we can evaluate the constant value of v by averaging it on Γ, which leads to $v=Ag$, $u=P(I+F)f+Ag$, and $g=J_0 f+Ag=J_0(f+Ag)$.

We can obtain the general result by applying this special case to the manifolds Γ_t for $t<0$. To this end, let $\boldsymbol{x}'$ and $\boldsymbol{y}'$ be on Γ_t, $-\tau\leqq t<0$, and let L_t, F_t, and J_t be the operators on $L^1(\Gamma_t)$ with kernels

$$
\begin{aligned}
L_t(\boldsymbol{x}',\boldsymbol{y}') &= c\,|\Gamma_t|^{-1}\int_{\Gamma_t}|\boldsymbol{x}'-\boldsymbol{y}'|^{1-k}d\sigma_{x'} \\
&\qquad +(1-k)c(\boldsymbol{x}'-\boldsymbol{y}')\cdot\boldsymbol{\nu}(\boldsymbol{x}')\,|\boldsymbol{x}'-\boldsymbol{y}'|^{-1-k} \\
F_t(\boldsymbol{x}',\boldsymbol{y}') &= -L_t(\boldsymbol{x}',\boldsymbol{y}')-\int_{\Gamma_t}L_t(\boldsymbol{x}',\boldsymbol{z}')F_t(\boldsymbol{z}',\boldsymbol{y}')\,d\sigma_{z'} \\
J_t(\boldsymbol{x}',\boldsymbol{y}') &= c\,|\boldsymbol{x}'-\boldsymbol{y}'|^{1-k}+c\int_{\Gamma_t}|\boldsymbol{x}'-\boldsymbol{z}'|^{1-k}F_t(\boldsymbol{z}',\boldsymbol{y}')\,d\sigma_{z'};
\end{aligned}
\tag{2}
$$

these are for Γ_t what L, F, and J_0 are for Γ. Then if $\Delta u=0$ in G, it follows that u and $\partial u/\partial n$ assume their boundary values on Γ_t uniformly, so that by our preliminary result we have on Γ_t

$$
u(\boldsymbol{x}')-|\Gamma_t|^{-1}\int_{\Gamma_t}u(\boldsymbol{x}')\,d\sigma_{x'}=\int_{\Gamma_t}J_t(\boldsymbol{x}',\boldsymbol{y}')\,(\partial u/\partial n)\,(\boldsymbol{y}')\,d\sigma_{y'}.
\tag{3}
$$

Now let $v_t(\boldsymbol{x})$ be such that $\int_{\Gamma_t}\phi(\boldsymbol{x}')\,d\sigma_{x'}=\int_\Gamma\phi(\boldsymbol{x}+t\boldsymbol{\nu})v_t(\boldsymbol{x})\,d\sigma_x$ for every continuous function ϕ. Then $v_t(\boldsymbol{x})\to 1$ uniformly in $\boldsymbol{x}$ as $t\to 0$. Since $\boldsymbol{\nu}(\boldsymbol{x})$ is the normal to Γ_t at $\boldsymbol{x}+t\boldsymbol{\nu}(\boldsymbol{x})$, $\partial u/\partial n$ on Γ_t is $D_t u(\boldsymbol{x}+t\boldsymbol{\nu}(\boldsymbol{x}))$; thus relation (3) on Γ_t can be written

$$
\begin{aligned}
&u(\boldsymbol{x}+t\boldsymbol{\nu}(\boldsymbol{x}))-|\Gamma_t|^{-1}\int_\Gamma u(\boldsymbol{x}+t\boldsymbol{\nu}))v_t(\boldsymbol{x})\,d\sigma \\
&\qquad =\int_\Gamma J_t(\boldsymbol{x}+t\boldsymbol{\nu}(\boldsymbol{x}),\boldsymbol{y}+t\boldsymbol{\nu}(\boldsymbol{y}))D_t u(\boldsymbol{y}+t\boldsymbol{\nu}(\boldsymbol{y}))v_t(\boldsymbol{y})\,d\sigma_y.
\end{aligned}
\tag{4}
$$

In order to obtain a limiting relation as $t\to 0-$, define the operator J^t on $L^1(\Gamma)$ by the kernel

$$
J^t(\boldsymbol{x},\boldsymbol{y})=J_t(\boldsymbol{x}+t\boldsymbol{\nu}(\boldsymbol{x}),\boldsymbol{y}+t\boldsymbol{\nu}(\boldsymbol{y}))v_t(\boldsymbol{y});
\tag{5}
$$

Define A^t on $L^1(\Gamma)$ by $A^t f = |\Gamma_t|^{-1} \int_\Gamma f v_t d\sigma$; and define u_t on Γ by $u_t(\boldsymbol{x}) = u(\boldsymbol{x} + t\boldsymbol{\nu}(\boldsymbol{x}))$. Then according to (4),

$$u_t - A^t u_t = J^t D_t u_t \text{ for } -\tau \leqq t < 0. \tag{6}$$

Now by Lemma 3, $\| J^t - J_0 \| \to 0$ as $t \to 0-$, and the hypothesis of Theorem 2 is that $\| D_t u_t - f \| \to 0$, where $\| \cdot \|$ denotes the norm of an operator on, or a function in, $L^p(\Gamma)$. It follows that $J^t D_t u_t \to J_0 f$ in $L^1(\Gamma)$, and that $u_t - A^t u_t$ does the same.

Thus it remains only to show that there is a function g in $L^p(\Gamma)$ such that $u_t \to g$; it will then follow that $A^t u_t \to Ag$, and $g - Ag = J_0 f$, or $g = J_0(f + Ag)$, as desired. To find g, consider first $I(t) = \int_\Gamma u_t$. Then $I'(t) = \int_\Gamma D_t u_t \to \int_\Gamma f$, so that $I'(t)$ is bounded on $-\tau \leqq t < 0$ and $\lim_{t\to 0-} I(t)$ exists. Now from (6), $\lim_{t \to 0-} |\Gamma|^{-1} \int_\Gamma (u_t - A^t u_t) = |\Gamma|^{-1} \int_\Gamma J_0 f$; we have just seen that $\lim \int_\Gamma u_t$ exists, so it follows that $\lim A^t u_t$ exists. Finally, we obtain that $u_t = A^t u_t + J^t D_t u_t$ converges in $L^1(\Gamma)$ to $g = \lim_{t \to 0-} A^t u_t + J_0 f$. This concludes the proof of Theorem 2.

2. More general boundary value problems. Theorems 1 and 2 are existence and uniqueness theorems respectively for the L^p formulation of the Neumann problem. In this section we consider more general first order boundary conditions, but require that $\partial u / \partial n$ exist on Γ in the L^p sense, for some $p > 1$. This is reasonable if the boundary operator is actually of first order at all points of Γ.

Specifically, we look for a function u such that

i) $\Delta u = 0$ in G,

β) ii) $D_t u(\boldsymbol{x} + t\boldsymbol{\nu})$ converges in $L^p(\Gamma)$ as $t \to 0-$, for some $1 < p < \infty$, and

iii) $Bu + Tu = h$ on Γ,

where B is a first order differential operator with continuous coefficients, defined in a neighborhood of Γ; T is a bounded operator on $L^p(\Gamma)$; and h is in $L^p(\Gamma)$. In order to apply the theory of [3], we shall assume Γ is of class C_3. Write $Bu = a\partial u/\partial n + Vu$, where a is a continuous function on Γ, and V is a continuous vector field on Γ. Suppose u is any solution of (β), and

denote the boundary value $\partial u/\partial n$ by f. Then, by Theorem 2, there is a g in $L^p(\Gamma)$ such that $u(\mathbf{x}+t\mathbf{v})\to g$ as $t\to 0-$, and $g=J_0(f+Ag)$. Writing ϕ for $f+Ag$, we have $\partial u/\partial n=\phi-A\phi$. Thus in the boundary condition $Bu+Tu=h$ we can replace u by $J_0\phi$, and obtain the equation $a\partial u/\partial n+Vu+Tu=a(\phi-A\phi)+VJ_0\phi+TJ_0\phi=h$. Now it is easy to show (as in [2], Section II D), that VP_0 is a $C_0{}^\infty$ singular integral operator on Γ with symbol $i\{V,\xi/|\xi|\}$, where V is thought of as a cross-section of the tangent bundle of Γ, ξ is a vector in the cotangent bundle of Γ, and $\{\ ,\ \}$ is the bilinear form relating the two. Moreover the operators F, TJ_0, and aA are completely continuous. Thus $H=aI+VP_0+(VP_0F+TJ_0-aA)$ is a $C_0{}^\infty$ singular integral operator with symbol $a+i\{V,\xi/|\xi|\}$; and the solutions u of the problem (β) are isomorphic to the solutions ϕ of the singular integral equation

$$H\phi=h \tag{7}$$

by the isomorphism $u=P(I+F)\phi$.

The difference between the number of orthogonality relations that must be satisfied by h to guarantee a solution of (β), and the dimension of the space of solutions of the associated homogeneous problem: $\Delta u=0$ in G and $Bu+Tu=0$ on Γ, is called the index of the problem (β). We obtain the following result.

THEOREM 3. *If $a+i\{V,\xi/|\xi|\}$ does not vanish at any point on the cotangent bundle of Γ, and the dimension of Γ is $\geqq 2$, then the index of the problem (β) is zero.*

Proof. With these assumptions, Theorem 3 of [3] asserts that the Fredholm alternative holds for the equation (7), and hence for the original boundary value problem. Q. E. D.

Remark 1. The hypothesis of Theorem 3 implies that a is non-vanishing on Γ, i.e. that B is nowhere tangential. If B is real, the condition that $a+i\{V,\xi/|\xi|\}$ vanishes nowhere is equivalent to the condition that B is nowhere tangential

Remark 2. The condition that $a+i\{V,\xi/|\xi|\}$ vanishes nowhere is equivalent to the condition that the boundary operator B covers the Laplacian, in the sense of Schechter (see [1]).

3. The lemmas. In this section $|||T|||$ denotes the supremum for $1\leqq p\leqq\infty$ of the norm of the operator T on $L^p(\Gamma)$. This norm makes the set

of operators T for which $|||\,T\,|||$ is finite a Banach algebra. The operators P_0 and P_t are described early in Section 1.

LEMMA 1. *$|||\,P_0 - P_t\,||| \to 0$ as $t \to 0$. If f is bounded, $P_t f \to P_0 f$ uniformly, and $P_0 f$ is continuous.*

Proof. The kernel of $P_0 - P_t$ is $c\,|\,\boldsymbol{x} - \boldsymbol{y}\,|^{1-k} - c\,|\,\boldsymbol{x} + t\boldsymbol{\nu}(\boldsymbol{x}) - \boldsymbol{y}\,|^{1-k} = cQ_t(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are points on Γ, and $|\,\boldsymbol{x} - \boldsymbol{y}\,|$ is the distance from $\boldsymbol{x}$ to $\boldsymbol{y}$ in E^{k+1}. Let $\rho(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{x} - \boldsymbol{y})/|\,\boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y}\,|$ $(\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}))$. Then a simple argument, based on the fact that $\boldsymbol{y}$ lies outside the sphere of radius τ about $\boldsymbol{x} \pm \tau\boldsymbol{\nu}$, shows that $0 \leqq \rho \leqq 2$ for $0 \leqq |\,t\,| \leqq \tau$. It follows from this that $|\,1 - \rho\,| \leqq 1$, and also $|\,1 - \rho\,| \leqq |\,t\,|/|\,\boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y}\,| \leqq 2\,|\,t\,|/|\,\boldsymbol{x} - \boldsymbol{y}\,|$. Thus for the kernel of $P_0 - P_t$ we have $cQ_t(\boldsymbol{x},\boldsymbol{y})$, and

$$|\,Q_t(\boldsymbol{x},\boldsymbol{y})\,| = |\,\boldsymbol{x} - \boldsymbol{y}\,|^{1-k}\,|\,1 - \rho^{k-1}\,| \leqq 2^{k-1}\,|\,\boldsymbol{x} - \boldsymbol{y}\,|^{1-k}\,|\,1 - \rho\,|,$$

or

$$\text{i)} \qquad |\,Q_t(\boldsymbol{x},\boldsymbol{y})\,| \leqq 2^{k-1}\,|\,\boldsymbol{x} - \boldsymbol{y}\,|^{1-k} \min(1, 2\,|\,t\,|/|\,\boldsymbol{x} - \boldsymbol{y}\,|).$$

Now let $\phi(s)$ be the characteristic function of $0 \leqq s \leqq 1$, and set $R_t(\boldsymbol{x},\boldsymbol{y}) = \phi(|\,\boldsymbol{x} - \boldsymbol{y}\,|\,|\,t\,|^{-\frac{1}{2}k})Q_t(\boldsymbol{x},\boldsymbol{y})$, and $S_t(\boldsymbol{x},\boldsymbol{y}) = Q_t(\boldsymbol{x},\boldsymbol{y}) - R_t(\boldsymbol{x},\boldsymbol{y})$. Then $S_t(\boldsymbol{x},\boldsymbol{y})$ vanishes for $|\,\boldsymbol{x} - \boldsymbol{y}\,| \leqq |\,t\,|^{\frac{1}{2}}$, so by (i)

$$|\,S_t(\boldsymbol{x},\boldsymbol{y})\,| \leqq 2^{k-1}\,|\,t\,|\,|\,\boldsymbol{x} - \boldsymbol{y}\,|^{-k} \leqq 2^{k-1}\,|\,t\,|^{\frac{1}{2}},$$

and $S_t(\boldsymbol{x},\boldsymbol{y})$ defines an operator S_t with $|||\,S_t\,||| = 0(|\,t\,|^{\frac{1}{2}})$. Using (i) again, we find that on each coordinate neighborhood $R_t(\boldsymbol{x},\boldsymbol{y})$ is dominated by a convolution kernel (in local coordinates), and this dominating kernel comes from a function in $L^1(E^k)$ whose norm is $0(|\,t\,|^{1/2k})$. Thus $|||\,P - P_t\,||| = 0(|\,t\,|^{1/2k})$ as $t \to 0$.

$P_t f$ is continuous for f in $L^1(\Gamma)$ and $t \neq 0$. If f is in L^∞ as well, the above result shows that $P_t f$ converges uniformly to $P_0 f$, and hence that $P_0 f$ is continuous. This establishes Lemma 1.

Remember that D_t denotes differentiation with respect to t.

LEMMA 2. *If f is in $L^p(\Gamma)$, $1 \leqq p < \infty$, then $D_t P_t f \to K_0 f + f$ in L^p norm, as $t \to 0-$, where $K_0 f(\boldsymbol{x}) = \int_\Gamma K_0(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, d\sigma_y$, and for $\boldsymbol{x} \neq \boldsymbol{y}$ $K_0(\boldsymbol{x},\boldsymbol{y}) = D_t c\,|\,\boldsymbol{x} + t\boldsymbol{\nu}(\boldsymbol{x}) - \boldsymbol{y}\,|^{1-k}$ evaluated at $t = 0$. $K_0(\boldsymbol{x},\boldsymbol{y})$ is uniformly $0(|\,\boldsymbol{x} - \boldsymbol{y}\,|^{1-k})$, and on $\Gamma \times \Gamma$ is a continuous function for $x \neq y$. If f is continuous, $D_t P_t f$ converges uniformly to $K_0 f + f$. If f is bounded, $K_0 f$ is continuous.*

Proof. For $t \neq 0$,

$$D_t P_t f(\boldsymbol{x}) = (1-k)c \int_\Gamma (\boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y}) \cdot \boldsymbol{\nu} \, | \boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y} |^{-k-1} f(\boldsymbol{y}) \, d\sigma_y,$$

where $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$ is the inner product of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in E^{k+1}. The kernel here is $K_t(\boldsymbol{x},\boldsymbol{y}) + B_t(\boldsymbol{x},\boldsymbol{y})$, with $K_t(\boldsymbol{x},\boldsymbol{y}) = (1-k)c(\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{\nu} \, | \boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y} |^{-k-1}$, and $B_t(\boldsymbol{x},\boldsymbol{y}) = (1-k)ct \, | \boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y} |^{-k-1}$. Since Γ lies between the spheres of radius τ with centers at $\boldsymbol{x} \pm \tau\boldsymbol{\nu}$, we have for $\boldsymbol{x}$ and $\boldsymbol{y}$ on Γ and $| \boldsymbol{x} - \boldsymbol{y} | < \tau$ that $| (\boldsymbol{x}-\boldsymbol{y}) \cdot \boldsymbol{\nu} | \leqq | \boldsymbol{x} - \boldsymbol{y} |^2/\tau$. Thus $K_t(\boldsymbol{x},\boldsymbol{y})$ is $0(| \boldsymbol{x} - \boldsymbol{y} |^{1-k})$ uniformly in $\boldsymbol{x}$, $\boldsymbol{y}$, and t, and an argument like that of Lemma 1 shows that if K_t and K_0 are the operators with kernels $K_t(\boldsymbol{x},\boldsymbol{y})$ and $K_0(\boldsymbol{x},\boldsymbol{y})$, then $||| K_t - K_0 ||| \to 0$ as $t \to 0$. As a consequence, if f is bounded then $K_0 f$ is continuous, as the uniform limit of the continuous functions $K_t f$.

We show next that $B_t(\boldsymbol{x},\boldsymbol{y})$ satisfies (i): for each $\delta > 0$, $B_t(\boldsymbol{x},\boldsymbol{y}) \to 0$ uniformly in $\boldsymbol{x}$ and $\boldsymbol{y}$, for $| \boldsymbol{x} - \boldsymbol{y} | > \delta$, as $t \to 0$; and (ii): $\int_\Gamma B_t(\boldsymbol{x},\boldsymbol{y}) \, d\sigma_y \to 1$ uniformly as $t \to 0-$ (and to -1 as $t \to 0+$). Claim (i) is clear, so we turn to (ii). Introduce orthonormal coordinates $(\xi_1, \cdots, \xi_k, \zeta)$ in E^{k+1}, with origin at the point $\boldsymbol{x}$ in Γ, so that $\zeta = 0$ is tangent to Γ at $\boldsymbol{x}$; and choose α so that $0 < \alpha < \tau$, and for $| \boldsymbol{x} - \boldsymbol{y} | < \alpha$ Γ can be represented by $\zeta = h(\xi_1, \cdots, \xi_k) = h(\boldsymbol{\xi})$. For simplicity of notation, we let $\boldsymbol{\xi}$ stand for the k-tuple such that $(\boldsymbol{\xi}, h(\boldsymbol{\xi})) = (\xi_1, \cdots, \xi_k, h(\boldsymbol{\xi}))$ are the coordinates of the point $\boldsymbol{y}$ in $\Gamma \cap \{ | \boldsymbol{x} - \boldsymbol{y} | < \alpha \}$. Then referring again to the two spheres about $\boldsymbol{x} \pm \tau\boldsymbol{\nu}$, we have the rough estimate

$$\begin{aligned}(\tau - | \boldsymbol{\xi} |)/(\tau + | \boldsymbol{\xi} |) \leqq | \boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y} |^2/(| \boldsymbol{\xi} |^2 + t^2) \\ \leqq (\tau + | \boldsymbol{\xi} |)/(\tau - | \boldsymbol{\xi} |),\end{aligned}$$

where $| \boldsymbol{\xi} |^2 = \sum_1^k \xi_i^2$. Also, in $\{ | \boldsymbol{x} - \boldsymbol{y} | < \alpha \} \cap \Gamma$ we have $d\sigma_y = v_x(\boldsymbol{\xi}) \, d\boldsymbol{\xi}$, and $v_x(\boldsymbol{\xi}) \to 1$ uniformly in $\boldsymbol{x}$, as $\boldsymbol{\xi} \to 0$. It is not hard to show that for $t < 0$, $(1-k)c \int_{E^k} t(| \boldsymbol{\xi} |^2 + t^2)^{-(k+1)/2} d\boldsymbol{\xi} = 1$, and hence that for any $\delta > 0$ $(1-k)c \int_{|\xi|<\delta} t(| \boldsymbol{\xi} |^2 + t^2)^{-(k+1)/2} d\boldsymbol{\xi} \to 1$, as $t \to 0-$. Thus for any positive $\eta < 1$ we can choose a positive $\delta < \alpha$ so that for all $\boldsymbol{x}$ and $| \boldsymbol{\xi} | < \delta$ we have $\eta^2 < | \boldsymbol{x} + t\boldsymbol{\nu} - \boldsymbol{y} |^2/(| \boldsymbol{\xi} |^2 + t^2) < \eta^{-2}$ and $\eta < v_x(\boldsymbol{\xi}) < \eta^{-1}$; and then we can choose $t < 0$ so small that $\eta < (1-k)c \int_{|\xi|<\delta} (| \boldsymbol{\xi} |^2 + t^2)^{-(k+1)/2} d\boldsymbol{\xi} \leqq 1$. Now let U_x be the neighborhood of $\boldsymbol{x}$ in Γ projecting onto $| \xi | < \delta$. Then $\int_{\Gamma - U_x} B_t(\boldsymbol{x},\boldsymbol{y}) \, d\sigma_y \to 0$ uniformly in $\boldsymbol{x}$ as $t \to 0$; and with $t < 0$ we have for

$$\int_{U_x} B_t(x,y)\,d\sigma_y = \int_{|\xi|<\delta} (1-k)ct\,|\,x + t\nu - y\,|^{-k-1} v_x(\xi)\,d\xi$$

that $\eta^{k+3} < \int_{U_x} B_t(x,y)\,d\sigma_y < \eta^{-k-2}$. Thus property (ii) of $B_t(x,y)$ is established.

Now $B_t(x,y)$ is clearly positive for $t < 0$, hence has the essential properties of an approximate identity. By a standard argument, $B_t f \to f$ uniformly for any continuous function f. Then another standard argument, relying on the fact that continuous functions are dense in $L^p(\Gamma)$ for $1 \leqq p < \infty$, shows that for f in L^p, $1 \leqq p < \infty$, $B_t f \to f$ in L^p norm. This, together with the results for K_t, establishes Lemma 2.

It is easy to see that $\lim_{t\to 0+} D_t P_t f = -f + K_0 f$. This with Lemma 2 is a formulation of the "jump" relations for the adjoint of the double layer potential.

LEMMA 3. *If J^t is the operator on $L^p(\Gamma)$ $(1 \leqq p \leqq \infty)$ defined by the krenel* (5), *then for each p $\|J^t - J_0\| \to 0$ as $t \to 0$.*

Proof. With reference to the kernels defined in (2), set

$$L^t(x,y) = L_t(x',y')v_t(y), \qquad F^t(x,y) = F_t(x',y')v_t(y),$$
$$\text{and } P^t(x,y) = c\,|\,x' - y'\,|^{1-k} v_t(y),$$

where $x' = x + t\nu(x)$, $y' = y + t\nu(y)$, and

$$\int_{\Gamma_t} \phi(y')\,d\sigma_{y'} = \int_\Gamma \phi(y + t\nu(y))\,v_t(y)\,d\sigma_y$$

for continuous ϕ. Then

$$F^t(x,y) = -L^t(x,y) - \int_\Gamma L^t(x,z)F^t(z,y)\,d\sigma_z$$

and $J^t(x,y) = P^t(x,y) + \int_\Gamma P^t(x,z)F^t(z,y)\,d\sigma_z$. We will achieve the desired result by showing in turn that $|||\,P^t - P_0\,||| \to 0$, $|||\,L^t - L\,||| \to 0$, $\|F^t - F\|_p \to 0$, and finally $\|J^t - J_0\|_p \to 0$. The first two cases are quite similar, and we discuss only $P^t - P_0$. The kernel of this operator is $c\{|\,x' - y'\,|^{1-k} v_t(y) - |\,x - y\,|^{1-k}\}$, which is $0(|x - y|^{1-k})$ uniformly in x, y, and t, and for $|\,x - y\,| \geqq \delta > 0$ converges uniformly to 0. Hence for any $\epsilon > 0$ we can choose $\delta > 0$ so that restricting the kernel to $|\,x - y\,| \leqq \delta$ yields an operator whose norm on $L^p(\Gamma)$, for all $1 \leqq p \leqq \infty$, is $< \epsilon/2$. We can then choose $t_0 > 0$ so that, for $|\,t\,| < t_0$, restricting the kernel to $|\,x - y\,| \geqq \delta$ yields

an operator whose norm on L^p, for all $1 \leqq p \leqq \infty$, is $< \epsilon/2$. Thus for $|t| < t_0$, $|||P^t - P_0||| < \epsilon$. In the same way, $|||L^t - L_0||| \to 0$ as $t \to 0$.

Since $I + L$ has trivial null space, and L is completely continuous on $L^p(\Gamma)$, $\phi + L\phi = \psi$ has a unique solution ϕ for each ψ in $L^p(\Gamma)$, i.e. $I + L$ is one to one and onto. For any fixed p in $1 \leqq p \leqq \infty$, it follows from the complete continuity of L that the inf $\| \phi + L\phi \|_p$ for $\| \phi \|_p = 1$ is not zero; for if $\| \phi_n \|_p = 1$ and $\phi_n + L\phi_n \to 0$, we can choose a subsequence ϕ'_n with $L\phi'_n$ convergent, from which it follows that for some ϕ $\phi'_n \to \phi$, with $\| \phi \|_p = 1$ and $\phi + L\phi = 0$. This contradicts the fact that $I + L$ has trivial null space. Thus $I + L$ has a bounded inverse and hence for sufficiently small t, $I + L^t$ has a bounded inverse that converges (in L^p operator norm) to $(I + L)^{-1}$, as $t \to 0$. Then $F^t = -(I + L^t)^{-1}L^t$ converges to $F = -(I + L)^{-1}$, and $J^t = p^t(I + F^t)$ converges to $J_0 = P_0(I + F)$, as required in the statement of the lemma.

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AFFINE RINGS OVER RANK TWO REGULAR LOCAL RINGS.* [1]

By LOUIS J. RATLIFF, JR.

1. Introduction. An integral domain $\mathfrak{o}$ is an *affine ring* over a subdomain I in case I is a Noetherian domain and $\mathfrak{o}$ is finitely generated over I. If $\mathfrak{o}$ is an affine ring over I and if $\mathfrak{p}$ is a prime ideal in $\mathfrak{o}$, then the quotient ring $P = \mathfrak{o}_{\mathfrak{p}}$ of $\mathfrak{o}$ with respect to $\mathfrak{p}$ is said to be a *spot* over I, hence a spot P is a (Noetherian) local domain. If the completion (relative to the powers of the product of the maximal ideals) of a (Noetherian) semi-local domain P contains no nonzero nilpotent elements, then P is said to be *analytically unramified.*

A Noetherian domain I is said to satisfy the *condition* (SF) if each separably generated affine ring $\mathfrak{o}$ over I is such that the integral closure of $\mathfrak{o}$ in its quotient field is a finite $\mathfrak{o}$-module. It is well known ([9], p. 267) that a field satisfies the condition (SF), and recently Nagata ([4], p. 419) has proven that a Dedekind domain satisfies this condition. Nagata's proof is essentially based on the following two facts.

1.1. Let $\mathfrak{o}$ be an affine ring over I, and let $\mathfrak{o}^*$ be an integral domain which contains $\mathfrak{o}$ and is a finite $\mathfrak{o}$-module. If $\mathfrak{p}^*$ is a prime ideal in $\mathfrak{o}^*$, then rank $\mathfrak{p}^* =$ rank $(\mathfrak{p}^* \cap \mathfrak{o})$.

1.2. If P is a separably generated spot over I, then the integral closure of P in its quotient field is a finite P-module ([4], p. 417).

It is known ([4], p. 414) that if a local domain P is analytically unramified, then the integral closure of P in its quotient field is a finite P-module. Hence statement 1.2 above is a consequence of

1.3. If P is a spot which is separably generated over I, then P is analytically unramified ([4], p. 416).

In this paper, generalizing the methods of Nagata [3, 4], it is proven that statements 1.1 and 1.3 remain valid if I is a rank two regular local ring which has an infinite residue field, and as a consequence it is proven

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that a rank two regular local ring which has an infinite residue field satisfies the condition (SF).

For the remainder of this section the letter I will denote a rank two regular local ring which has an infinite residue field.

Section 2 contains the definitions of most of the terms used in this paper. By employing some results of Nagata [5, 6] it is proven in Section 3 that if $\mathfrak{o}$ is an affine ring over I and if $\mathfrak{q}$ is a maximal ideal in $\mathfrak{o}$, then the transcendence degree of $\mathfrak{o}$ over I is equal to the rank of $\mathfrak{q}$ minus the rank of $\mathfrak{q} \cap I$ (in symbols, $\operatorname{trd}(\mathfrak{o}/I) = \operatorname{rank} \mathfrak{q} - \operatorname{rank}(\mathfrak{q} \cap I)$). Several corollaries to this theorem which are needed in Section 4 are then proven, and Corollary 2 is the generalization of statement 1.1 above to the case where I is a rank two regular local ring. In Section 4 it is proven that statement 1.3 above remains true when I is a rank two regular local ring which has an infinite residue field. The proof is broken into three parts. Namely, let P be a spot which is separably generated over I. If P does not dominate I, then P is a spot which is separably generated over a Dedekind domain or a field, hence P is analytically unramified by Nagata's result mentioned in 1.3 above. Secondly, if P dominates I and contains a set of elements which has a certain property ("property (T)") defined below, then by means of a finite number of quadratic transforms of I it is shown that P is a subspace of a semi-local domain P^* such that, if M^* is a maximal ideal in P^*, then $P^*_{M^*}$ is a spot which is separably generated over a Dedekind domain. Hence, by certain well known theorems on completions of a semi-local domain ([10], pp. 277, 283), P is analytically unramified. Finally, if P dominates I and does not contain a set of elements which has the property (T), then it is necessary to derive some information concerning the structure of such a spot (Remark 4.2 and Lemma 4.5). These structure theorems are generalizations of two theorems ([3], p. 86, and [4], p. 397) due to Nagata, and are used to show that there is imbedded in P a semi-local domain D such that P is a spot over D and D has the same quotient field as P. Then by using a recent result due to Rees [8] it is seen that P is analytically unramified by proving that D is analytically unramified. Having proven that every spot P which is separably generated over I is analytically unramified, it follows that the integral closure of P in its quotient field is a finite P-module ([4], p. 414), and then Nagata's proof that a Dedekind domain has the property (SF) carries over verbatim to prove that a rank two regular local ring which has an infinite residue field has the property (SF).

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2. Definitions and notation. All rings in this paper are assumed to be commutative rings with a unit element 1.

A ring $\mathfrak{o}$ has *rank* r in case there exists a chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r \tag{2.1}$$

of $r+1$ proper prime ideals $\mathfrak{p}_i$ in $\mathfrak{o}$, and there does not exist such a chain with more than $r+1$ prime ideals. (The symbol $\subset$ is used to indicate *proper* inclusion). If $\mathfrak{p}$ is a prime ideal in $\mathfrak{o}$, then the *rank* of $\mathfrak{p}$ is the rank of $\mathfrak{o}_{\mathfrak{p}}$, and if $\mathfrak{a}$ is an ideal in $\mathfrak{o}$, then the *rank* of $\mathfrak{a}$ is the minimum of the ranks of the prime divisors of $\mathfrak{a}$. A chain (2.1) of prime ideals in an integral domain $\mathfrak{o}$ is a *maximal chain* of prime ideals in $\mathfrak{o}$ in case the rank of $\mathfrak{p}_i/\mathfrak{p}_{i-1}$ is one $(i=1,\cdots,r)$, $\mathfrak{p}_0=(0)$, and $\mathfrak{p}_r$ is a maximal ideal in $\mathfrak{o}$. The *length* of the chain is r.

Let $\mathfrak{o}$ be an integral domain. The integral closure $\mathfrak{o}'$ of $\mathfrak{o}$ in its quotient field is the *derived normal ring* of $\mathfrak{o}$, and $\mathfrak{o}$ is said to be *normal* in case $\mathfrak{o}$ is equal to $\mathfrak{o}'$.

If a ring P has only one maximal ideal M, then this fact will often be displayed by the notation (P,M), and it is said that (P,M) is a *quasi-local ring*. Let (P,M) and (Q,N) be two quasi-local rings. Then P *dominates* $Q (P \geqq Q)$ in case $P \supseteq Q$ and $M \cap Q = N$. If a quasi-local ring P is a Noetherian ring, then P is called a *local ring*. A Noetherian ring P which has only a finite number of maximal ideals is called a *semi-local ring*.

Remark 2.1. Let P, Q, and R be local rings such that P is a spot over Q which dominates Q, and Q is a spot over R which dominates R. Then P is a spot over R which dominates R.

Let (P,M) be a local ring, let $y_1,\cdots,y_t$ be algebraically independent over P, and let $R=P[y_1,\cdots,y_t]$. Then the ring R_{MR} will be denoted by $P(y_1,\cdots,y_t)$.

A ring I is a *ground ring* in case I is a field, a Dedekind domain, or a rank two regular local ring which has an infinite residue field. The theorems which will be proven in this paper are known [3, 4] if the ground ring is a field or a Dedekind domain. Therefore *all future statements in this paper concerning a ground ring I are to be interpreted as statements about a rank two regular local ring which has infinitely many elements in its residue field.*

Let (P,M) be a spot over a ground ring (I,m). A ring B contained in P is said to be a *basic ring* in P in case 1) P is a spot over B, and B is a spot over I, 2) $P \geqq B$, and $B \geqq I_{(M\cap I)}$, 3) B is a field, a Dedekind domain, or a ground ring, and 4) P/M is a finite algebraic extension of $B/(M\cap B)$.

Remark 2.2. Let (P, M) be a spot over a ground ring (I, m). If P contains a basic ring (B, N), then P is a quotient ring with respect to a *maximal* ideal of an affine ring over B.

Proof. By the definition of a basic ring $P = A_{\mathfrak{q}}$, where $A = B[b_1, \cdots, b_n]$ and $\mathfrak{q}$ is a prime ideal in A. Suppose that $\mathfrak{q}$ is *properly* contained in $\mathfrak{q}^*$, where $\mathfrak{q}^*$ is a maximal ideal in A. Since $P \geqq B$, $\mathfrak{q} \cap B = N$, and since $\mathfrak{q}^* \supset \mathfrak{q}$, $\mathfrak{q}^* \cap B = N$. Hence the natural homomorphism of $(B/N)[b_1', \cdots, b_n']$ onto $(B/N)[b_1^*, \cdots, b_n^*]$ has a nonzero kernel, where b_i' and b_i^* are the residue classes of b_i modulo $\mathfrak{q}$ and $\mathfrak{q}^*$ respectively. Therefore $\operatorname{trd}((A/\mathfrak{q})/(B/N)) > \operatorname{trd}((A/\mathfrak{q}^*)/(B/N))$. But B is a basic ring in P and P/M is isomorphic to the quotient field of $A/\mathfrak{q}$, so $\operatorname{trd}((A/\mathfrak{q})/(B/N)) = \operatorname{trd}((P/M)/(B/N)) = 0$, hence $\operatorname{trd}((A/\mathfrak{q}^*)/(B/N)) < 0$, contradiction. Hence $\mathfrak{q}$ is a maximal ideal in A, q. e. d.

Let (R, N) be a local domain, and let (S, M) be a spot over R which dominates R. Then R satisfies the *dimension formula* in case

$$(2.2) \qquad \operatorname{rank} S + \operatorname{trd}((S/M)/(R/N)) = \operatorname{rank} R + \operatorname{trd}(S/R).$$

An integral domain R satisfies the *first chain condition* in case every maximal chain of prime ideals in R has length equal to the rank of R. R satisfies the *second chain condition* in case every integral domain which contains R and is integrally dependent on R satisfies the first chain condition.

A *derivation* D of an integral domain S is an additive endomorphism of the quotient field K of S which satisfies the conditions 1) $D(xy) = xDy + yDx$, for all $x, y \in K$, and 2) there exists an element d in S such that $dDx \in S$, for all $x \in S$. If $Da = 0$ for every element a in a subdomain R of S, then D is said to be a *derivation of S over R*.

3. The rank of an affine ring over a ground ring. As stated in Section 2 the statement that I is a ground ring is to mean that I is a rank two regular local ring which has an infinite residue field. However, the assumption that the residue field of a ground ring I is infinite is not needed in this section.

Nagata [5, 6] has proven two theorems which are stated as Remark 3.1 for future reference.

Remark 3.1. The second chain condition holds in a Noetherian domain I if and only if the first chain condition holds in the derived normal ring of I. Further, if I satisfies the second chain condition, and if P is a spot over I, then P satisfies the second chain condition and the dimension formula.

LEMMA 3.1. *Let (I, m) be a ground ring, and let $\mathfrak{p}$ be a prime ideal in I. Then $I_{\mathfrak{p}}$ satisfies the dimension formula and the second chain condition.*

Proof. Since $(0) \subset m$ is not a maximal chain of prime ideals in I, I satisfies the first chain condition. Since I is a normal Noetherian domain ([10], p. 302), I satisfies the second chain condition (Remark 3.1). Since $I_{\mathfrak{p}}$ is a spot over I, $I_{\mathfrak{p}}$ satisfies the dimension formula and the second chain condition (Remark 3.1), q. e. d.

THEOREM 3.1. *Let $\mathfrak{o} = I[z_1, \cdots, z_n]$ be an affine ring over a ground ring (I, m), let $\mathfrak{q}$ be a maximal ideal in $\mathfrak{o}$, and let $\mathfrak{q} \cap I = \mathfrak{p}$. Then* $\operatorname{trd}(\mathfrak{o}/I) = \operatorname{rank} \mathfrak{q} - \operatorname{rank} \mathfrak{p}$.

Remark. It is not difficult to show that there does not exist a maximal ideal $\mathfrak{q}$ in $\mathfrak{o}$ such that $\mathfrak{q} \cap I = (0)$. Hence, if $\mathfrak{q}$ is a maximal ideal in $\mathfrak{o}$, then $\operatorname{rank} \mathfrak{q} = r + 1$ or $r + 2$, where $r = \operatorname{trd}(\mathfrak{o}/I)$. Since no use is made of this fact in the remainder of this paper, the proof that $\mathfrak{q} \cap I \neq (0)$ will not be given.

Proof of Theorem 3.1. Set $P = \mathfrak{o}_{\mathfrak{q}}$ and $M = \mathfrak{q}P$, so (P, M) is a spot over $I_{\mathfrak{p}}$. Since $I_{\mathfrak{p}}$ satisfies the second chain condition (Lemma 3.1), every maximal chain of prime ideals in P has length equal to the rank of P (Remark 3.1), hence equal to the rank of M. Since there is a one to one correspondence between the prime ideals in $\mathfrak{o}$ which are contained in $\mathfrak{q}$ and the prime ideals in P, every maximal chain of prime ideals in $\mathfrak{o}$ which ends at $\mathfrak{q}$ has length equal to the rank of $\mathfrak{q}$. By Remark 3.1 $I_{\mathfrak{p}}$ satisfies the dimension formula (2.2), so $\operatorname{rank} P + \operatorname{trd}((\mathfrak{o}/\mathfrak{q})/(I/\mathfrak{p})) = \operatorname{rank} I_{\mathfrak{p}} + \operatorname{trd}(\mathfrak{o}/I)$, since $\operatorname{trd}((P/M)/(I_{\mathfrak{p}}/\mathfrak{p}I_{\mathfrak{p}})) = \operatorname{trd}((\mathfrak{o}/\mathfrak{q})/(I/\mathfrak{p}))$, and $\operatorname{trd}(\mathfrak{o}/I) = \operatorname{trd}(P/I_{\mathfrak{p}})$. Further, $\mathfrak{o}$ is finitely generated over I and $\mathfrak{q}$ is a maximal ideal in $\mathfrak{o}$, hence $\operatorname{trd}((\mathfrak{o}/\mathfrak{q})/(I/\mathfrak{p})) = 0$. Therefore

$$\operatorname{trd}(\mathfrak{o}/I) = \operatorname{rank} P - \operatorname{rank} I_{\mathfrak{p}} = \operatorname{rank} \mathfrak{q} - \operatorname{rank} \mathfrak{p},$$

q. e. d.

COROLLARY 3.1. *With the same notation as in Theorem 3.1, if $(0) = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_s = \mathfrak{q}$ is a maximal chain of prime ideals in $\mathfrak{o}$, then* $s = \operatorname{rank} \mathfrak{q}$.

Proof. Proved in the course of proving Theorem 3.1.

COROLLARY 3.2. *Let $\mathfrak{o}$ be an affine ring over a ground ring I, let $\mathfrak{o}^*$ be an integral domain which contains $\mathfrak{o}$ and is a finite $\mathfrak{o}$ module, and let $\mathfrak{p}^*$ be a prime ideal in $\mathfrak{o}^*$. Then* $\operatorname{rank} \mathfrak{p}^* = \operatorname{rank}(\mathfrak{p}^* \cap \mathfrak{o})$.

Proof. Consider a maximal chain of prime ideals through $\mathfrak{p}^*$ in $\mathfrak{o}^*$ ending at, say, $\mathfrak{q}^*$. The length of this chain is equal to the rank of $\mathfrak{q}^*$ (Corollary 3.1). Since $\mathfrak{o}^*$ is integrally dependent on $\mathfrak{o}$, the contraction in $\mathfrak{o}$ of the ideals in this chain yields a chain of distinct prime ideals in $\mathfrak{o}$, and $\mathfrak{q}^* \cap \mathfrak{o}$ is a maximal ideal in $\mathfrak{o}$. Clearly $\operatorname{trd}(\mathfrak{o}^*/I) = \operatorname{trd}(\mathfrak{o}/I)$, and $\mathfrak{q}^* \cap I = (\mathfrak{q}^* \cap \mathfrak{o}) \cap I$, so $\operatorname{rank}(\mathfrak{q}^* \cap I) = \operatorname{rank}((\mathfrak{q}^* \cap \mathfrak{o}) \cap I)$, hence $\operatorname{rank} \mathfrak{q}^* = \operatorname{rank}(\mathfrak{q}^* \cap \mathfrak{o})$ (Theorem 3.1). Since the contracted chain of prime ideals has length equal to the rank of $\mathfrak{q}^*$, it has length equal to the rank of $\mathfrak{q}^* \cap \mathfrak{o}$, hence it is a maximal chain of prime ideals in $\mathfrak{o}$. Therefore $\operatorname{rank}(\mathfrak{p}^* \cap \mathfrak{o}) = \operatorname{rank} \mathfrak{p}^*$, q. e. d.

COROLLARY 3.3. *Let (P, M) be a spot over a ground ring I, let $A = P[\theta_1, \cdots, \theta_h]$ be a finite P-module, and let $\mathfrak{p}$ be a prime ideal in A. Then* $\operatorname{rank} \mathfrak{p} = \operatorname{rank}(\mathfrak{p} \cap P)$.

Proof. Let Q be a maximal ideal in A which contains $\mathfrak{p}$, and let $P = \mathfrak{o}_{\mathfrak{q}}$, where $\mathfrak{q}$ is a prime ideal in $\mathfrak{o} = I[z]$ $(= I[z_1, \cdots, z_n])$. Since A is a finite P-module, each θ_i is integrally dependent on P, hence there exist elements $r_i \in \mathfrak{o}$, $r_i \notin \mathfrak{q}$, such that the elements $r_i\theta_i$ are integrally dependent on $\mathfrak{o}$. Set $S = \mathfrak{o}[r\theta]$, so S is a finite $\mathfrak{o}$-module. Clearly $A \supseteq S$, so

$$A_Q \geqq S_{(Q \cap S)} \geqq \mathfrak{o}_{(Q \cap \mathfrak{o})} = \mathfrak{o}_{\mathfrak{q}} = P,$$

hence $S_{(Q \cap S)} \supseteq A$, since the elements $r_i \notin Q \cap S$. Therefore $A_Q = S_{(Q \cap S)}$. Since S is a finite $\mathfrak{o}$-module, and since $(\mathfrak{p} \cap S) \cap \mathfrak{o} = (\mathfrak{p} \cap P) \cap \mathfrak{o}$, $\operatorname{rank}(\mathfrak{p} \cap S) = \operatorname{rank}((\mathfrak{p} \cap P_1) \cap \mathfrak{o})$ (Corollary 3.2). Since A_Q and P are quotient rings of S and $\mathfrak{o}$ respectively, $\operatorname{rank}(\mathfrak{p} \cap S) = \operatorname{rank} \mathfrak{p}$, and $\operatorname{rank}((\mathfrak{p} \cap P) \cap \mathfrak{o} = \operatorname{rank}(\mathfrak{p} \cap P)$, q. e. d.

COROLLARY 3.4. *Let (P, M) be a spot over a ground ring I. Assume that P contains a basic ring (B, M), where B is a rank two regular local ring which dominates I. Then* $\operatorname{rank} P = \operatorname{trd}(P/B) + 2$.

Proof. By Remark 2.2 P is a quotient ring with respect to a maximal ideal of an affine ring over B. Since B is a ground ring, and since $\operatorname{rank}(M \cap B) = \operatorname{rank} N = 2$, $\operatorname{rank} P = \operatorname{trd}(P/B) + 2$ (Theorem 3.1), q. e. d.

4. Analytical unramifiedness of spots which are separably generated over a ground ring. The main part of this section is devoted to proving

THEOREM 4.1 *If (P, M) is a spot which is separably generated over a ground ring (I, m), then P is analytically unramified.*

The proof of this theorem is attained by combining propositions 4.1, 4.2, and 4.3 below.

PROPOSITION 4.1. *Let (P, M) be a spot which is separably generated over (I, m). If P does not dominate I, then P is analytically unramified.*

Proof. Since P does not dominate I, the rank of $M \cap I$ is one or zero. Since a quotient ring with respect to a prime ideal of a regular local ring is a regular local ring ([7], p. 208), and since a rank one regular local ring is a local Dedekind domain, if rank $M \cap I = 1$, then $I_{(M \cap I)}$ is a local Dedekind domain, and if $M \cap I = (0)$, then $I_{(M \cap I)}$ is a field. Therefore P is a spot which is separably generated over a Dedekind domain or a field, hence P is analytically unramified ([4], p. 416), q. e. d.

Henceforth *the statement that P is a spot over I will mean that P is a spot which is separably generated over a ground ring I and which dominates I.* Further, *the letter t will consistently be used to denote the transcendence degree of P/M over I/m.*

Definition 4.1. Let (P, M) be a spot over (I, m). A set of t elements $y_1, \cdots, y_t$ in P is said to have the *property* (T) in case the residues of the y_i modulo M are a transcendence base for P/M over I/m, and the y_i are algebraically dependent over I.

Definition 4.2. Let (P, M) be a spot over (I, m). If P contains a set of element which has the property (T), then P is said to be a spot of *type one.* P is a spot of *type two* in case P is not a spot of type one.

COROLLARY 4.1. *If (P, M) is a rank one spot over (I, m), then P is a spot of type one.*

Remark. There exist rank one spots which dominate I. For example, let $m = (a, b)I$, let $R = I[b/a]$, and let $\mathfrak{p}$ be a rank one prime divisor of aR. Then $R_{\mathfrak{p}}$ is a rank one spot which dominates I.

Proof of Corollary 4.1. By Remark 3.1 I satisfies the dimension formula (2.2), so $t = \mathrm{trd}((P/M)/(I/m)) = 1 + \mathrm{trd}(P/I)$, hence every set of elements in P whose residues modulo M are a transcendence base for P/M over I/m is a set of element which has the property (T), hence P is a spot of type one, q. e. d.

LEMMA 4.1. *Let (P, M) be a spot of type one over (I, m), let $y_1, \cdots, y_t$ be a set of elements in P which has the property (T), and let $S = I[y_1, \cdots, y_t]$. Then mS is a prime ideal in S, and P dominates S_{mS}.*

Proof. Let $P = \mathfrak{o}_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal in $\mathfrak{o} = I[z]$ $(= I[z_1, \cdots, z_n])$. Let σ be the restriction to S of the natural homomorphism from P onto P/M. Since $P \geqq I$, P/M is isomorphic to $(I/m)(z_1', \cdots, z_n')$, $S/(M \cap S)$ is isomorphic to $(I/m)[y_1', \cdots, y_t']$ (the $'$ indicates residues modulo M), and $\ker \sigma = M \cap S \supseteq mS$. Conversely, if $f(y)$ is in $\ker \sigma$, then all coefficients of f are in m, since the y_i' are algebraically independent over I/m. Thus $\ker \sigma = mS$, so mS is a prime ideal, hence $P \geqq S_{(M \cap S)} = S_{mS}$, q. e. d.

LEMMA 4.2. *Let (P, M) be a spot over (I, m), let P' be the derived normal ring of P, let $M_1', \cdots, M_h'$ be the totality of maximal ideals in P', let θ_i be elements in P' such that $\theta_i \equiv \delta_{ij}$ (M_j') $(i, j = 1, \cdots, h)$, and let $A = P[\theta_1, \cdots, \theta_h]$. Then A is a semi-local domain which has exactly h maximal ideals $Q_i = M_i' \cap A$. Let $P_i = A_{Q_i}$. Then the derived normal ring P_i' of P_i is $P'_{M_i'}$.*

Remark. The fact that P' has only a finite number of maximal ideals is proven in [2].

Proof. Set $Q_i = M_i' \cap A$. Since A is integrally dependent on P, the derived normal ring A' of A is equal to P', hence each Q_i is a maximal ideal in A, and the Q_i are distinct by the choice of the θ_i. Therefore there exist at least h maximal ideals in A. If Q is an arbitrary maximal ideal in A, then there exists a maximal ideal M' in P' such that $M' \cap A = Q$, since P' is integrally dependent on A. Since M' is one of the M_i', $Q = M_i' \cap A = Q_i$. Thus there exist exactly h maximal ideals in A. Also A is Noetherian, since P is Noetherian, hence A is a semi-local domain. Fix i and set $R = P'_{M_i'}$, $N = M_i'R$, and $\mathfrak{p} = N \cap P_i'$ (R contains P_i', since R is normal). Then $R \geqq (P_i')_{\mathfrak{p}}$. If $\mathfrak{q}$ is a maximal ideal in P_i', then $\mathfrak{q} \cap A' = \mathfrak{q} \cap P' = M_i'$, so $(P_i')_{\mathfrak{q}} \geqq R \geqq (P_i')_{\mathfrak{p}}$, hence $\mathfrak{p} = \mathfrak{q}$, and $P_i' = (P_i')_{\mathfrak{p}} = R$, q. e. d.

Remark 4.1. With the same notation as in Lemma 4.2, if $d_1, \cdots, d_k$ are elements in P_i', then $S = P_i[d_1, \cdots, d_k]$ is a *spot* over I, and rank S = rank P. Further, $\operatorname{trd}((S/Q)/(I/m)) = \operatorname{trd}((P_i/M_i)/(I/m))$, where Q and M_i are the unique maximal ideals in S and P_i respectively.

Proof. The proof that S is a spot over I follows from Remark 2.1 and the fact that P_i' is a quasi-local domain, and rank S = rank P by Corollary 3.3. The last statement is clear, since the d_j are integrally dependent on P_i.

LEMMA 4.3. *Let (P, M) be a spot of type one over (I, m). There exists a semi-local domain P^* which is contained in the derived normal ring*

of P such that 1) *P^* contains P and is a finite P-module, and* 2) *if M^* is a maximal ideal in P^*, then $P^*_{M^*}$ is a spot which is separably generated over a Dedekind domain.*

Proof. Let P' be the derived normal ring of P, and in the notation of Lemma 4.2 let $A=P[\theta_1,\cdots,\theta_h]$. Then by Lemma 4.2 A is a semi-local domain which has h maximal ideals $Q_i=M_i'\cap A$, P_i' is a quasi-local domain, and by Remark 4.1 P_i is a spot over I $(i=1,\cdots,h)$. Let $y_1,\cdots,y_t$ be a set of elements in P which has the property (T). Since the θ_i are integrally dependent on P, it is clear that $y_1,\cdots,y_t$ are elements in P_i which form a set which has the property (T). Further, if an element d is in the derived normal ring of P_j, then there exists an element $r\in A$, $r\notin Q_j$ such that rd is an element in the derived normal ring of A. Hence, in view of Remark 4.1 it is sufficient to prove the lemma in the case where the derived normal ring of P is a quasi-local domain (and in this case the ring P^* is a *local* domain).

Let L and F be the quotient fields of P and I respectively, let (R_v,M_v) be the valuation ring of a real discrete valuation v of L such that $R_v\geqq P$ (existence of R_v is proven in [1]), let w be the restriction of v to F, and let (T_w,N_w) be the valuation ring of w. Then clearly $T_w\geqq I$, hence w is not trivial on F. Let $Y_1,\cdots,Y_t$ be indeterminants, and extend w to $I[Y]$ $(=I[Y_1,\cdots,Y_t])$ by defining $w(g(Y))=\min\{w(r_i)\,;\ r_i$ is a coefficient of $g(Y)\}$. Since $y_1,\cdots,y_t$ are elements in P which form a set which has the property (T), there exist polynomials $f(Y)$ in $I[Y]$ such that $f(y)=0$ and reducing modulo M the equation $f(y)=0$ it is seen that all coefficients of $f(Y)$ are in m. Among the polynomials $f(Y)$ such that $f(y)=0$ let $g(Y)$ be such that $w(g(Y))$ is a minimum (such a polynomial exists since $T_w\geqq I$). Let $g(Y)=\sum r_iM_i(Y)$, where $M_i(Y)$ is a monomial with a coefficient which is a unit in I. Then $w(g(Y))>0$, so it may be assumed tat each $r_i\in m^\beta$, and that at least one $r_i\notin m^{\beta+1}$. Let $m=(a,b)I$. Then $r_i=\sum r_{ij}a^{\beta-j}b^j$, and since at least one $r_i\notin m^{\beta+1}$, at least one r_{ij} is a unit in I. Set $\phi_j(Y)=\sum r_{ij}M_i(Y)$, so $g(Y)=\sum\phi_j(Y)a^{\beta-j}b^j$. Further, $w(\phi_j(Y))=0$, for some j, hence $\phi_j(y)\neq 0$.

Under the unimodular transformation $b_1=b$, $a_1=a-cb_1$, where c is a unit in I, the coefficient of $b_1{}^\beta$ in $g(Y)$ is $\sum\phi_j(Y)c^{\beta-j}$. Since I/m is an infinite field, there exist values of c such that the coefficient of $b_1{}^\beta$ $(=\sum_i(\sum_j r_{ij}c^{\beta-j})M_i(Y))$, is such that for at least one $i\sum r_{ij}c^{\beta-j}$ is a unit in I, hence $\sum\phi_j(y)c^{\beta-j}\neq 0$. Thus it may be assumed that the coefficient $\phi_\beta(Y)$ of b^β in $g(Y)$ has the property that $w(\phi_\beta(Y))=0$, hence $\phi_\beta(Y)$ has a coefficient which is a unit in I. Therefore $\phi_\beta(y)$ is a unit in S_{mS}, where

$S=I[y_1,\cdots,y_t]$ (mS is a prime ideal by Lemma 4.1). By Lemma 4.1 $P\geqq S_{mS}$, hence $\phi_\beta(y)$ is a unit in P. Hence, from

$$0=g(y)=\phi_\beta(y)b^\beta+\phi_{\beta-1}(y)b^{\beta-1}a+\cdots+\phi_0(y)a^\beta,$$

it is seen that b/a is integrally dependent on P.

Set $P_0=P$, $P_1=P_0[b/a]$, and let M_1 be the maximal ideal in P_1. Since P' is a quasi-local domain, P_1 is a *local* domain contained in P' which is a finite P-module and which dominates P. Set $I_0=I$, $J=I_0[b/a]$, $I_1=J_{(M_1\cap J)}=J_{(N_w\cap J)}$, and $m_1=(M_1\cap J)I_1$. Then (P_1,M_1) is a spot (which is separably generated) over (I_1,m_1), and $P_1\geqq I_1$. (I_1 is called the *first quadratic transform* of I along w. It is known ([4], p. 404) that I_1 is a regular local ring which dominates I, and rank $I_1\leqq$ rank I. Let I_k be the first quadratic transform of I_{k-1} ($k\geqq 1$) along w (assuming rank $I_{k-1}>1$). Then I_k is called the *k-th quadratic transform* of I along w). If rank $I_1=1$, then the theorem is proved. If rank $I_1=2$, then $y_1,\cdots,y_t$ are elements in P_1 which form a set with the property (T), $R_v\geqq P_1$, $T_w\geqq I_1$, and the derived normal ring of P_1 is P', so the above process can be repeated. Let $g_1(Y)$ be an element in $I_1[Y]$ such that $g_1(y)=0$ and $w(g_1(Y))$ is a minimum. Then $w(g_1(Y))>0$. Since $r_i\in m^\beta\subseteq a^\beta I_1$, $r_i=a^\beta\mu_i$, where $\mu_i\in I_1$, so $g(Y)=a^\beta f(Y)$, where $f(Y)=\sum\mu_iM_i(Y)$. Since $g(y)=0$, $f(y)=0$, hence $w(g(Y))>w(f(Y))\geqq w(g_1(Y))>0$. Since $w(g(Y))$ is an integer, in a finite number of steps, say k, rank $I_k=1$, where I_k is the k-th quadratic transform of I along w, and P_k is a spot (which is separably generated) over I_k. Since I_k is a regular local ring of rank one, I_k is a local Dedekind domain, q. e. d.

PROPOSITION 4.2. *Let (P,M) be a spot over (I,m). If P is a spot of type one, then P is analytically unramified.*

Proof. Let P^* be the semi-local domain constructed in Lemma 4.3, and let M^* be a maximal ideal in P^*. Then $P^*_{M^*}$ is a spot which is separably generated over a Dedekind domain, hence $P^*_{M^*}$ is analytically unramified ([4], p. 416). Since the completion of P^* is isomorphic to the direct sum of the completions of the $P^*_{M^*}$ ([10], p. 283), P^* is analytically unramified Since P^* is an integral domain which is a finite P-module (Lemma 4.3), the completion of P is a subring of the completion of P^* ([10], p. 277), hence P is analytically unramified, q. e. d.

COROLLARY 4.2. *Let (P,M) be a rank one spot over (I,m). Then P is analytically unramified.*

Proof. By Corollary 4.1 P is a spot of type one, q. e. d.

Let (P, M) be a spot of type two over (I, m). The *every* set of t elements $y_1, \cdots, y_t$ in P which map modulo M onto a transcendence base for P/M over I/m are algebraically independent over I, hence the ring $B = I(y_1, \cdots, y_t)$ is a basic ring in P. (B is a rank two regular local ring, since the y_i are algebraically independent over I). Since P is a spot over I, $P = \mathfrak{o}_\mathfrak{q}$, where $\mathfrak{q}$ is a prime ideal in an affine ring $\mathfrak{o}$ over I. Since P/M is isomorphic to the quotient field of $\mathfrak{o}/\mathfrak{q}$, there exist $y_1, \cdots, y_t \in \mathfrak{o}, \notin \mathfrak{q}$ such that the y_i map modulo M onto a transcendence base for P/M over I/m.

Definition 4.3. *Let $P = \mathfrak{o}_\mathfrak{q}$ be a spot of type two over (I, m).* A basic ring B in P is a *special basic ring* in P in case $B = I(y_1, \cdots, y_t)$, where $y_1, \cdots, y_t$ are elements in $\mathfrak{o}$, not in $\mathfrak{q}$, whose residues modulo $\mathfrak{q}$ are a transcendence base for $\mathfrak{o}/\mathfrak{q}$ over I/m.

Remark 4.2. If P is a spot of type two over I, then P contains a special basic ring.

Remark 4.3. Every special basic ring in P is a rank two regular local ring.

LEMMA 4.4. *Let $(P, M) = (\mathfrak{o}_\mathfrak{q}, \mathfrak{q}\mathfrak{o}_\mathfrak{q})$ be a spot of type two over (I, m), let* $\operatorname{rank} P = r$, *and let (B, mB) be a special basic ring in P, where $B = I(y_1, \cdots, y_t)$. Then there exist $r - 2$ elements in $\mathfrak{q}$ which are algebraically independent over $A = I[y_1, \cdots, y_t]$.*

Proof. Since P is a spot of type two, $r \geqq 2$ (Corollary 4.1). Since $\operatorname{rank} P = \operatorname{trd}(P/B) + 2$ (Corollary 3.4), $\operatorname{trd}(0/A) = r - 2$. If $r = 2$, then there is nothing to prove, so it may be assumed that $r > 2$. Hence there exists an element $z_1 \in \mathfrak{o}$ which is transcendental over A, and it may be assumed that $z_1 \notin \mathfrak{q}$. Then the residue z_1' of z_1 modulo $\mathfrak{q}$ is algebraic over $A/(\mathfrak{q} \cap A)$ (by the choice of the y_i). Let $f_1'(z_1') = 0$ be a (not necessarily monic) minimal equation of z_1' over $A/(\mathfrak{q} \cap A)$, and let $f_1(z_1) = x_1 \neq 0$ be an element in $\mathfrak{q}$ which maps modulo $\mathfrak{q}$ onto $f_1'(z_1')$. Since z_1 is transcendental over A, x_1 is transcendental over A. If $r = 3$, then the lemma is proved, so it may be assumed that $r > 3$. Then there exists an element $z_2 \in \mathfrak{o}$ such that z_2 is transcendental over $A[x_1]$, and it may be assumed that $z_2 \notin \mathfrak{q}$. As above let $f_2(z_2) = x_2$ be an element in $\mathfrak{q}$ which maps modulo $\mathfrak{q}$ onto a minimal equation $f_2'(z_2') = 0$ of z_2' over $A/(\mathfrak{q} \cap A)$. Then $x_2 \neq x_1$, $x_2 \neq 0$, and since z_2 is transcendental over $A[x_1]$, x_2 is transcendental over $A[x_1]$, hence x_1 and x_2 are algebraically independent over A. Hence in $r - 2$ steps the required elements are obtained, q. e. d.

LEMMA 4.5. *Let (P,M) be a spot of type two over (I,m), and let* $\operatorname{rank} P = r$ $(r \geqq 2)$. *Then there exists a special basic ring B^* in P and elements $z_1,\cdots,z_{r-2}$ in M which are algebraically independent over B^* such that P is a finite separable algebraic extension of $B^*[z_1,\cdots,z_{r-2}]$.*

Remark. The proof of this lemma is a direct generalization of Nagata's proof of Proposition 8 ([4], p. 397).

Proof. Since P is separably generated over I, when the characteristic of I is 0 the lemma is true by Lemma 4.4, so it may be assumed that the characteristic of I is $p \neq 0$. Let $P = \mathfrak{o}_{\mathfrak{q}}$, and let $B = I(y_1,\cdots,y_t)$ be a special basic ring in P (B exists by Remark 4.2). Let $A = I[y_1,\cdots,y_t]$, and let $x_1,\cdots,x_{r-2}$ be elements in $\mathfrak{q}$ which are algebraically independent over A (Lemma 4.4). Then $\operatorname{trd}(\mathfrak{o}/A[x_1,\cdots,x_{r-2}]) = 0$, hence $\operatorname{trd}(\mathfrak{o}/I) = t + (r-2)$. Let $n = t + r - 2$, let $w_i = x_i$ $(i = 1,\cdots,r-2)$, and let $w_{r-2+i} = y_i$ $(i = 1,\cdots,t)$. Since P is separably generated over I, let $D_1,\cdots,D_n$ be a maximal linearly independent set of derivations of $\mathfrak{o}$ over I (existence is proven in ([4], p. 390)). Let $u_1,\cdots,u_n$ be elements in $\mathfrak{o}$ such that $D_i u_j \neq 0$ if and only if $i = j$. It may be assumed that $u_j - 1 \in \mathfrak{q}$ $(j = 1,\cdots,n)$, since otherwise the elements $1 + u_j\alpha^p$ $(\alpha \in \mathfrak{q})$ are such that $D_i(1 + u_j\alpha^p) \neq 0$ if and only if $i = j$, and $(1 + u_j\alpha^p) - 1$ are in $\mathfrak{q}$. Set $z_j = w_j^p u_j$. Then $D_i z_j \neq 0$ if and only if $i = j$, hence $z_1,\cdots,z_n$ are a separating transcendence base for $\mathfrak{o}$ over I. Since $u_j - 1 \in \mathfrak{q}$, $z_j' = w_j'^p u_j' = w_j'^p$ (where the $'$ indicates residues modulo q). Now $w_j'^p = 0$ $(j = 1,\cdots,r-2)$, and the $w_j'^p$ $(j = r-1,\cdots,n)$ are a transcendence base for $\mathfrak{o}/\mathfrak{q}$ over I/m. Therefore the ring $B^* = I(z_{r-1},\cdots,z_n)$ is a special basic ring in P, and the elements $z_1,\cdots,z_{r-2}$ are elements in $\mathfrak{q}$ which are algebraically independent over B^*, q. e. d.

PROPOSITION 4.3. *Let (P,M) be a spot over (I,m). If P is a spot of type two, then P is analytically unramified.*

Proof. Let $\operatorname{rank} P = r$ $(r \geqq 2)$. Let (B, mB) be a special basic ring in P such that there exist elements $x_1,\cdots,x_{r-2}$ in M such that P is a finite separable algebraic extension of $C^* = B[x_1,\cdots,x_{r-2}]$ (Lemma 4.5). Set $Q^* = (mB, x_1,\cdots,x_{r-2})C^*$, $C = C^*_{Q^*}$, and $Q = Q^*C$. Since B is a regular local ring of rank two (Remark 4.3), and since the z_i are algebraically independent over B (Lemma 4.5), (C,Q) is a regular localy ring of rank r, and $P \geqq C$. Let E be the quotient field of C, and let L be the quotient field of P, so L is a finite separable algebraic extension of E. Hence, since C is a normal Noetherian domain, the integral closure C' of C in L is a finite

C-module ([9], pp. 264-265). Therefore the integral closure D of C in P is a finite C-module. Since P is a spot over B, there exists elements $d_1, \cdots, d_g \in P$ such that P is a quotient ring with respect to a prime ideal of the ring $C[d_1, \cdots, d_g]$, and since the d_i are algebraic over E, there exist elements $s_i \in C$ such that the elements $s_i d_i \in D$, hence the quotient field of D is L. Since $P \geqq D_{(M \cap D)} \geqq C$, $M \cap D$ is a maximal ideal in D and there exist elements $e_1, \cdots, e_k \in L$ such that P is a quotient ring with respect to a prime ideal of $D_{(M \cap D)}[e_1, \cdots, e_k]$. Hence by a recent result due to Rees [8], P is analytically unramified if $D_{(M \cap D)}$ is analytically unramified. Since D is a finite C-module, D is a semi-local domain, hence $D_{(M \cap D)}$ is analytically unramified if D is ([10], p. 283), and since C' is a semi-local domain which is a finite D-module, D is analytically unramified if C' is ([10], p. 277). Hence to prove that P is analytically unramified it is sufficient to prove that C' is analytically unramified.

Since L is a finite separable algebraic extension of E, let u be an element in C' such that $L = \sum_{i=0}^{\alpha} Eu^i$, and set $H = \sum_{i=0}^{\alpha} Cu^i$. Then H is an integral domain which contains C and is contained in C', hence H is a semi-local domain with quotient field L. Let F be the total quotient ring of the completion of H. Since C' is a finite C-module, there exists a nonzero element $d \in C$ such that $dC' \subseteq H$. Since the completion of C is a regular local ring ([10], p. 302), d is not a zero divisor in the completion of H ([10], p. 277), hence F contains the completion of C'. Let K be the quotient field of the completion of C. Then it is seen that $F = \sum_{i=0}^{\alpha} Ku^i$. Since F is a finite dimensional algebra over K which contains the completion of C' and is contained in the total quotient ring of the completion of C', F is the total quotient ring of the completion of C'. It is a straight forward verification that F is isomorphic to $L \otimes_E K$, and since L is separably generated over E, $L \otimes_E K$ contains no nonzero nilpotent elements ([9], p. 195). Therefore C' is analytically unramified, hence P is analytically unramified, q. e. d.

Propositions 4.1, 4.2, and 4.3 constitute a proof of Theorem 4.1.

It is known ([4], p. 414) that if a semi-local domain P is analytically unramified, then the derived normal ring of P is a finite P-module, hence

COROLLARY 4.3. *If P is a spot which is separably generated over a ground ring I, then the derived normal ring of P is a finite P-module.*

As mentioned in the introduction, Nagata's proof that a Dedekind domain has the property (SF) is essentially based on statements 1.1 and 1.2. Since

Corollary 3.2 and Corollary 4.3 are respectively the generalizations of these statements to the case where I is a rank two regular local ring which has an infinite residue field, Nagata's proof carries over verbatim to prove

THEOREM 4.2. *If $\mathfrak{o}$ is an affine ring which is separably generated over a ground ring I, then the derived normal ring of $\mathfrak{o}$ is a finite $\mathfrak{o}$-module.*

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UNIFORMLY BOUNDED REPRESENTATIONS, II.*

Analytic Continuation of the Principal Series of Representations of the $n \times n$ Complex Unimodular Group.

By R. A. Kunze[1] and E. M. Stein.[2]

0. Introduction. This paper is the second of a series devoted to the study of uniformly bounded representations of classical groups and the application of these representations to analysis on these groups.

In the previous paper [8], this was carried out for the special case of the 2×2 real unimodular group. Two related results were basic. The first dealt with the analytic continuation of the 'principal series' of representations —leading to a family of uniformly bounded representations defined in a strip. The second result applied the analytic family of representations to obtain a 'Hausdorff-Young Theorem' for the group together with its implications for the characterization of representations.

Our purpose here is to carry out the analytic continuation of the principal series in the case of the group of $n \times n$ complex unimodular matrices. The problems dealing with the continuation for other semi-simple groups and the problems connected with the Hausdorff-Young Theorem will be considered in future papers of this series.

We consider the group G of $n \times n$ complex unimodular matrices, and its diagonal (i.e., Cartan) sub-group C, whose entries we denote by $c = (c_1, c_2, \cdots, c_n)$, $c \in C$. Let λ denote a continuous character of the subgroup C. Then

$$\lambda(c) = \left(\frac{c_1}{|c_1|}\right)^{m_1} \left(\frac{c_2}{|c_2|}\right)^{m_2} \cdots \left(\frac{c_n}{|c_n|}\right)^{m_n} \cdot |c_1|^{s_1} \cdots |c_n|^{s_n} \tag{0.1}$$

where $s_1, s_2, \cdots, s_n$ are complex numbers and $m_1, m_2, \cdots, m_n$ are integers; both sequences are uniquely determined if we require that $s_1 + s_2 + \cdots + s_n = 0$ and $0 \leqq m_1 + m_2 + \cdots + m_n < n$. (We call the integer r, $r = m_1 + m_2 + \cdots + m_n$, $0 \leqq r < n$, the *residue* of the character λ.) Notice that the character λ is unitary if $Re(s_j) = 0$, $j = 2, \cdots, n$.

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Gelfand and Neumark have shown how to construct an irreducible unitary representation for each unitary character λ. This leads to the 'principal series' of representations; these are the representations which arise in the Plancherel formula for the group. They are constructed as follows.

For each such λ the representation $a \rightarrow T(a, \lambda)$ is realized as a multiplier representation on $L_2(V)$, where V is the sub-group of G consisting of matrices which have ones on the main diagonal and whose entries above the main diagonal vanish. The representation $T(a, \lambda)$ is defined by

$$T(a, \lambda): f(v) \rightarrow K(va, \lambda) f(v\bar{a}) \tag{0.2}$$

where K is an appropriate multiplier and the mapping $v \rightarrow v\bar{a}$ may be thought of as a generalization of fractional linear transformations which occur when $n = 2$.

Now the multiplier appearing in (0.2) depends on λ and thus on $s_1, s_2, \cdots, s_n$. This dependence may in fact be seen to be analytic. Thus we can obtain in a trivial sense an analytic continuation of (0.2). This naive analytic continuation is, however, not the right one for a variety of reasons. The simplest of these is that the resulting operators are not uniformly bounded, in fact not everywhere defined on $L^2(V)$ when $Re(s_j) \neq 0$.*

The natural analytic continuation may be motivated by the following considerations.

We quote two known facts about the representations (0.2), (when $Re(s_j) = 0$). The first is that they already irreducible on a certain subgroup G_0 of G. The second fact is that $T(\cdot, \lambda)$ and $T(\cdot, p\lambda)$ are unitarily equivalent representations for any character λ and any permutation p of the character (i.e., element of the Weyl group of G).

It is convenient to group the characters λ into n distinct classes—(their residue classes). The class to which any character belongs is determined by the unique integer r, $0 \leqq r < n$, for which $m_1 + m_2 + m_3 + \cdots + m_n = r$. Or put another way, the class is determined by the action of λ on the center of G.

We come now to what is in effect a basic result of this paper. The

* It is possible to bypass the last objection and obtain by analytic continuation everywhere defined (but not uniformly bounded) representations by reformulating the representations (0.2) in terms of the maximal compact sub-group. See Harish-Chandra, "Representations of a semi-simple Lie group I," *Trans. Amer. Math. Soc.*, vol. 75 (1953), p. 241. It should be added, however, that in this sense there are infinitely many "analytic continuations" of the principal series. Only by adding certain natural considerations can one find a unique analytic continuation, which in addition, enjoys the further properties which are described hereafter.

representations $T(\cdot,\lambda_1)$ and $T(\cdot,\lambda_2)$ for which λ_1 and λ_2 belong to the same class are equivalent when restricted to G_0. This is remarkable in view of the fact that they are already irreducible on G_0. Thus there exists a unitary operator, $W(\lambda)$, so that if we define $R(a,\lambda)$ by

$$R(a,\lambda) = W(\lambda)T(a,\lambda)W^{-1}(\lambda) \tag{0.3}$$

and if we fix $a \in G$, then $R(a,\lambda)$ depends only on the class of λ (and thus in particular is constant as a function of $s_1, s_2, \cdots, s_n$). In view of the already quoted irreducibility of $T(\cdot,\lambda)$ restricted to G_0, the representations $R(\cdot,\lambda)$ so determined are unique. We call these the *normalized principal series.* It is these representations that can be continued analytically in the parameters of λ while $T(a,\lambda)$ and $W(\lambda)$ separately can not.

The discussion above insures the existence of $W(\lambda)$ and thus $R(a,\lambda)$ *a priori,* but tells us nothing about their concrete properties which are necessary for analytic continuation. We need first, therefore, construct $W(\lambda)$ in explicit terms.

In order to get an idea of how to go about this, we recall, as mentioned above, that $T(\cdot,\lambda)$ and $T(\cdot,p\lambda)$ are unitarily equivalent; therefore $R(\cdot,\lambda)$ and $(R\cdot,p\lambda)$ are unitarily equivalent. Using the fact that $R(a,\lambda) = R(a,p\lambda)$ for $a \in G_0$, it follows (from the irreducibility on G_0) that $R(a,\lambda) = R(a,p\lambda)$ for all $a \in G$, and finally $W^{-1}(p\lambda)W(\lambda)T(a,\lambda) = T(a,p\lambda)W^{-1}(\lambda)W(p\lambda)$ for all p in the Weyl group.

This shows that an explicit determination of $W(\lambda)$ brings with it a similarly explicit description of the intertwining operators $A(p,\lambda)$ defined by $A(p,\lambda)T(a,\lambda) = T(a,p\lambda)A(p,\lambda)$.

We reverse the heuristic argument described above and begin our study by obtaining the intertwining operators $A(p,\lambda)$ in a concrete fashion. It turns out that this task is much simplified when we restrict ourselves to the $n-1$ permutations $p_1, p_2, \cdots, p_{n-1}$, where p_r corresponds to the transposition $(r, r+1)$ of the symmetric group on n letters. These $n-1$ permutations generate the whole Weyl group. It is then found that the operators $A(p_r,\lambda)$ can be given by singular convolution integrals whose kernels are carried on one-dimensional (complex) sub-groups of V (see Theorem 1, Section 4).

A basic property of the operators $A(p,\lambda)$ is that although they do not commute in general, they satisfy certain fundamental commutativity relations (see Theorem 2, Section 5). These commutativity relations are a reflection of the relations satisfied by the generators $p_1, p_2, \cdots, p_{n-1}$ of the Weyl group. The actual analysis of the commutativity relations is rather complex but is simplified by reducing the problem to the case of the 3×3 group by a change

of variables. In this sense the 3×3 group is already the general case, because here we must deal with the non-commutativity of the sub-group $V(3)$, which makes the Fourier analysis on $L_2(V)$ difficult.

Once the properties of the operators $A(p,\lambda)$ are obtained, we can then define the operators $W(\lambda)$. These are given by products of $\frac{n(n-1)}{2}$ operators $A(p,\lambda)$, (see 6.5) and 5.17)). We then prove in Theorem 3 (Section 6) the basic properties of the normalized prinicipal series for unitary λ.

Our next task is the analytic continuation of this series. The problem of analytic continuation is tractable because of the following two considerations. 1) It suffices, in effect, to continue one parameter only keeping the others fixed. That this allows us to obtain analytic continuation simultaneously in all parameters follows from the lemmas of Section 8. 2) The group G is generated by the sub-group G_0 and a fixed element $p \in G$. Since the $R(a,\lambda)$, $a \in G_0$, have trivial (i. e., constant) analytic continuations, it suffices to consider the analytic continuation of the single operator $R(p,\lambda)$. Here use is made of the lemmas of Section 7.

The resulting uniformly bounded representations are defined when the parameters $s_1, s_2, \cdots, s_n$ vary over an open 'tube' in the complex hyperplane $s_1 + s_2 + \cdots + s_n = 0$. Their basic properties are set forth in Theorem 4 of Section 9.

In Section 10 we calculate the 'trace' of these representations which allows us to prove in particular the following: (i) $R(\cdot,\lambda)$ is unitary if and only if $\lambda' = p(\lambda)$, where λ' is the complex contragredient character of λ, and p is some element of the Weyl group. Hence $R(\cdot,\lambda)$ is unitary if and only if it corresponds to an element of the principal or complementary series. (ii) If $\lambda' \neq p(\lambda)$, then although $R(\cdot,\lambda)$ is uniformly bounded, it is not equivalent to a unitary representation.

1. The principal series. In this section we recall relevant facts about G and the construction of the principal series. Detailed proofs of most of the results we state below may be found in the book by Gelfand and Neumark [3].[3]

Let $GL = GL(n)$ denote the group of all complex non-singular matrices of degree n. In what follows we shall need to consider the following sub-groups of GL.

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[3] In stating the results of Gelfand and Neumark we have found it convenient to use a notation which is different from theirs.

$G = G(n)$: the unimodular sub-group.

$C = C(n)$: the (Cartan) sub-group of diagonal matrices $c = [c_1, c_2, \cdots, c_n]$ in G.

$U = U(n)$: the upper triangular unipotent matrices

$$u = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ \cdot & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}$$

$u_{jk} = 0$ for $j > k$ and $u_{jj} = 1$.

$V = V(n)$: the lower triangular unipotent matrices

$$v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdot \\ v_{n1} & \cdots & v_{nn-1} & 1 \end{bmatrix}$$

$v_{jk} = 0$ for $j < k$ and $v_{jj} = 1$.

For each a in GL, let

$$\begin{pmatrix} j_1 & \cdots & j_m \\ k_1 & \cdots & k_m \end{pmatrix}$$

be the determinant of the submatrix of a formed from rows $j_1, \cdots, j_m$ and columns $k_1, \cdots, k_m$ of a. Put

$$D_j(a) = \begin{pmatrix} j & j+1 & \cdots & n \\ j & j+1 & \cdots & n \end{pmatrix} \tag{1.1}$$

$$D_{n+1}(a) = 1.$$

Then in order that a may be expressed as a product

$$a = ucv, \quad u \in U, \quad v \in V, \quad c \text{ diagonal} \tag{1.2}$$

it is necessary and sufficient that $D_j(a) \neq 0$ for $1 \leqq j \leqq n$. The matrices u, v, and c are then uniquely determined as follows: Let

$$D_{jk}(a) = \begin{pmatrix} j & j+1 & \cdots & n \\ j & k+1 & \cdots & n \end{pmatrix}, \quad j < k \tag{1.3}$$

$$D_{jk}(a) = \begin{pmatrix} j & j+1 & \cdots & n \\ j & k+1 & \cdots & n \end{pmatrix}, \quad j > k. \tag{1.4}$$

Then u, v, and c are given by

$$u_{jk} = \frac{D_{jk}(a)}{D_k(a)}, \quad j < k \tag{1.5}$$

$$v_{jk} = \frac{D_{jk}(a)}{D_j(a)}, \quad j > k \tag{1.6}$$

$$c_j = \frac{D_a(a)}{D_{j+1}(a)}, \quad 1 \leqq j \leqq n. \tag{1.7}$$

If $a \in GL$ and all the determinants $D_j(a)$ are different from 0, we shall say that a is *decomposable*. The diagonal matrix c in (1.2) will be called the diagonal part of a and will be denoted by $\operatorname{diag}(a)$. It should be noted that $\operatorname{diag}(a)$ belongs to C, i.e., has determinant 1 if and only if a is in G.

The elements v of V are parametrized by the real and imaginary parts, x_{jk}, y_{jk} of their entries v_{jk}. The measure dv obtained by taking the product of the differentials

$$dv_{jk} = dx_{jk} dy_{jk}, \quad j > k \tag{1.8}$$

is both left and right invariant; so V is unimodular.

In the 2×2 case, V may be identified with the additive group of complex numbers, and GL then acts on V as the group of linear fractional transformations. In the general case GL acts on V in a similar fashion. We shall now describe this action. Let a be a fixed member of GL. Then for v in V, the determinants $D_j(va)$ are polynomial functions of the entries v_{jk} of v. Thus, with the exception of a lower dimensional set of measure 0, the product va is decomposable; the exceptional set of elements v depends on a. If va can be expressed as a product $u'c'v'$ with $u' \in U$, $v' \in V$, and c' diagonal, then this can be done in only one way and we shall put

$$v\tilde{a} = v'. \tag{1.9}$$

In the 2×2 case, $v\tilde{a}$ is the element of V defined by

$$(v\tilde{a})_{21} = \frac{v_{21}a_{11} + a_{21}}{v_{21}a_{12} + a_{22}}. \tag{1.10}$$

When n is arbitrary, there are still some instances in which $v\tilde{a}$ is easily described. For example, if c is diagonal, $v\tilde{c} = c^{-1}vc$, and for any $v_1 \in V$, $v\tilde{v}_1 = vv_1$.

Let $v \in V$, $a \in GL$, and suppose va is decomposable. If b is any member of GL, then vab is decomposable if and only if $(v\tilde{a})b$ is; when this is the case it is then true that

$$v(\widetilde{ab}) = (v\tilde{a})\tilde{b}. \tag{1.11}$$

When $n=2$, the mapping $v \to v\tilde{a}$ induces a transformation of the real parameters corresponding to v_{21} whose Jacobian is equal to

$$|\det a|^2 \cdot |v_{21}a_{12}+a_{22}|^{-4}.$$

Thus the measure $d(v\tilde{a})$ is given by

$$d(v\tilde{a}) = |\det a|^2 \cdot |v_{21}a_{12}+a_{22}|^{-4}\, dv. \tag{1.12}$$

In the general case, the measure $d(v\tilde{a})$ is obtained as follows: Let μ be the character of the diagonal sub-group of GL given by

$$\mu(c) = \prod_{j=1}^{n} |c_j|^{2(n-2j+1)}. \tag{1.13}$$

For each decomposable a in GL, let

$$\mu(a) = \mu(c)$$

where c is the diagonal part of a in the decomposition (1.2). Then with a fixed, $\mu(va)$ as a function of v is defined on all but a set of measure 0 in V; and

$$d(v\tilde{a}) = \mu(va)\,dv. \tag{1.14}$$

This is proved in [3] for the case in which a has determinpant 1. The general case may be deduced from this with the aid of the following observations. (i) If $a \in GL$ and b is a non-zero scalar multiple of a then a and b determine the same transformation of V; furthermore, (1.6) and (1.7) show that the digaonal parts of va and vb differ by a multiplicative factor. (ii) if $a \in GL$ then $(\det a)^{-1/n}a \in G$. (iii) In (1.13) the sum of the exponents is equal to 0.

If λ is any character of the diagonal sub-group of GL its domain of definition may be extended as above. Thus if $a=ucv$ with $u \in U$, $v \in V$, and c diagonal we shall put

$$\lambda(a) = \lambda(c). \tag{1.15}$$

After this extension λ is no longer multiplicative but has the following properties. If c is diagonal and a is decomposable, then ca and ac are decomposable, and

$$\lambda(ca) = \lambda(ac) = \lambda(c)\lambda(a). \tag{1.16}$$

If $v \in V$ then $\lambda(v)=1$, and

$$\lambda(vab) = \lambda(va)\lambda[(v\tilde{a})b] \tag{1.17}$$

whenever va and vab are both decomposable.

We shall say that a character of C is *unitary* if it maps C into the multiplicative group of complex numbers of absolute value 1. To each continuous unitary character λ of C is associated a unitary representation $a \to T(a, \lambda)$ of G on the Hilbert space, $L_2(V)$ of square integrable functions on V. These representations are known as the *principal series* and are defined for f in $L_2(V)$ by

$$T(a,\lambda)f(v) = \lambda(va)\mu^{\frac{1}{2}}(va)f(v\tilde{a}). \tag{1.18}$$

2. The decomposition $V = ZZ'$. In this section we decompose V into the product of two sub-groups Z and Z' and study the action of G on V in terms of this decomposition.

Let l and m be integers such that $1 \leqq l < m \leqq n$ and $m - l < n - 1$. For each $a \in G$, let $\dot{a}$ be the $n \times n$ matrix specified by the following conditions.

(i) $\dot{a}_{jk} = a_{jk}, \; l \leqq j, k \leqq m.$

(ii) $\dot{a}_{kk} = 1, \; k < l$ or $k > m$.

(iii) $\dot{a}_{jk} = 0$ in the cases not covered by (i) or (ii).

We shall denote by $\hat{a}$ the submatrix of a formed from the entries a_{jk}, $l \leqq j$, $k \leqq m$. The mapping $v \to \dot{v}$, $v \in V$ is a homomorphism of V onto a subgroup $Z = Z(l,m)$; Z is contained in V and is isomorphic to $V(m-l+1)$ via the mapping $z \to \hat{z}$. Let $Z' = Z(l,m)'$ be the kernel of the homomorphism $v \to \dot{v}$. The $n \times n$ identity matrix is the only element of V which is common to both Z and to Z'. Moreover, an easy computation shows that $\dot{v}^{-1}v \in Z'$ for each v in V. Thus to each $v \in V$ there correspond unique elements z and z' such that

$$v = zz', \; z \in Z, \; z' \in Z'. \tag{2.1}$$

To illustrate we consider the case $n = 4$. A typical element of $Z(2,4)$ has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & v_{32} & 1 & 0 \\ 0 & v_{42} & v_{43} & 1 \end{bmatrix}$$

whereas a typical element of $Z(2,3)'$ has the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ v_{21} & 1 & 0 & 0 \\ v_{31} & 0 & 1 & 0 \\ v_{41} & v_{42} & v_{43} & 1 \end{bmatrix}.$$

We shall denote by dz the invariant measure in Z obtained from the product of the differentials (1.8)

$$dz_{jk} = dx_{jk}dy_{jk} \qquad l \leqq k < j \leqq m.$$

Since Z' is a closed normal sub-group of V, it follows ([9; p. 133]) that Z' is unimodular and also ([9; p. 119]) that

$$dv = dz\, dz' \tag{2.2}$$

where dz' is a suitably chosen invariant measure on Z'. We shall not need the explicit form of dz' (actually, here the most obvious candidate for dz' works).

If $b = \dot{a}$ is non-singular, the mapping $z \to z\bar{b}$ is well defined and carries Z into Z. The transformed measure $d(z\bar{b})$ is obtained as follows: Let $\mu_{l,m}$ be thet character of C defined by

$$\mu_{l,m} = \prod_{j=l}^{m} | c_j |^{2(m+1-2j)}. \tag{2.3}$$

It then follows from (1.13) and (1.14) applied to the case $n = m - l + 1$ that

$$d(z\bar{b}) = \mu_{l,m}(ab)\, dz. \tag{2.4}$$

For certain a in G, $b = \dot{a}$ is non-singular, and the mapping $z \to z\bar{a}$, $z \in Z$ coincides with the mapping $z \to z\bar{b}$. We shall be interested in two sub-groups of G whose elements have this property. One of the sub-groups is the diagonal sub-group C. For suppose $c \in C$. Then $z\bar{c} = c^{-1}zc$ for each z in Z, and since

$$(c^{-1}zc)_{jk} = c_j^{-1} z_{jk} c_k$$

it is evident that c and $\dot{c}$ induce the same mapping of Z into Z. The other sub-group we shall call $U^+ = U^+(l, m)$. It contains U and is obtained as follows: Let $X = X(l, m)$ be the sub-group of G consisting of all x such that $x = \dot{x}$. The mapping $x \to \hat{x}$, $x \in X$ is an isomorphism between X and $G(m - l + 1)$. In addition, denote by $Y = Y(l, m)$ the kernel of the homomorphism $u = \dot{u}$, $u \in U$. It is then easily seen that $X \cap Y = \{1_n\}$, that X is in the normalizer of Y in G, and that the product, XY is a sub-group of G. We shall put $U^+ = XY$. To illustrate we again consider the case $n = 4$. A typical element of $U^+(2, 3)$ has the form

$$\begin{bmatrix} 1 & U_{12} & U_{13} & U_{14} \\ 0 & U_{22} & U_{23} & U_{24} \\ 0 & U_{32} & U_{33} & U_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let us now determine in general how an element of U^+ acts on Z. To do this we ask when an element xy of U^+ is decomposable. Since $xyx^{-1} \in U$ and $xy = xyx^{-1} \cdot x$ it follows that xy is decomposable if and only if x is decomposable; furthermore, the elements c and v in the decomposition (1.2) of xy depend only on x. On the other hand, since X is isomorphic to $G(m-l+1)$ it is evident that c and v are members of X and in fact that $v \in Z$. From these observations we obtain the following result.

LEMMA 1. *Let $z \in Z$, $x \in X$, and $y \in Y$. Then zxy is decomposable if and only if zx is decomposable. If zx and zxy are decomposable then*

$$\widetilde{zxy} = z\tilde{x} \in Z \tag{1}$$

$$\operatorname{diag}(zxy) = \operatorname{diag}(zx) \in X. \tag{2}$$

3. The operators $A(p, \lambda)$. In this section we shall use the decompositions $V = ZZ'$ in defining a class of operators $A(p, \lambda)$ in $L_2(V)$. These operators play a crucial role in our work on the representations of G. In order to understand something about their group theoretic significance it will be necessary to introduce some terminology.

Let k be an integer such that $1 \leqq k \leqq n-1$, and $Z = Z(k, k+1)$. Let $p = p_k$ be the element of $X = X(k, k+1)$ which satisfies

$$\hat{p} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.\text{[4]} \tag{3.1}$$

Suppose in addition that $m_1, m_2, \cdots, m_n$ are integers, that $s_1, s_2, \cdots, s_n$ are complex numbers, and that λ is the character of C given by

$$\lambda(c) = \prod_{j=1}^{n} \left(\frac{c_j}{|c_j|}\right)^{m_j} |c_j|^{s_j}. \tag{3.2}$$

We shall now obtain an explicit expression for the function $\lambda(zp)$, $z \in Z$. If $z \in Z$ then

$$\hat{z} = \begin{bmatrix} 1 & 0 \\ z_{k+1k} & 1 \end{bmatrix}.$$

Hence the mapping $z \to z_{k+1k}$ is an isomorphism between Z and the additive group of complex numbers; to simplify the notation we shall also denote z_{k+1k} by z. We use this convention in the statement of the following lemma.

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[4] For further clarification of the role of the matrices p_k see the discussion at the beginning of Section 4 and in Section 6 following equation (6.4).

LEMMA 2. *Suppose $z \in Z$, $u \in U$, and that λ is defined by* (3.2). *Then for $z \neq 0$,*

$$\lambda(zup) = \lambda(zp) = \left(\frac{z}{|z|}\right)^{m_{k+1}-m_k} |z|^{s_{k+1}-s_k}.$$

Proof. Since $U^+ = XY$ we may write $u = xy$, $x \in X$, $y \in Y$ (actually $x = \dot{u}$). Thus $zup = zxyp = zxp(p^{-1}yp)$. Because Y is normal in U^+, $p^{-1}yp \in Y$, and it follows from Lemma 1 that zup is decomposable if and only if zxp is decomposable.

In addition we have

$$\hat{x} = \hat{u} = \begin{bmatrix} 1 & u_{kk+1} \\ 0 & 1 \end{bmatrix}.$$

Now if we also write x for u_{kk+1} it follows that

$$\hat{z}\hat{x}\hat{p} = \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -x & 1 \\ -zx-1 & z \end{bmatrix}.$$

Hence if $z \neq 0$,

$$zxp = \begin{bmatrix} 1 & \dfrac{1}{z} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1}{z} & 0 \\ 0 & z \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \dfrac{zx+1}{-z} & 1 \end{bmatrix}.$$

From this we see that zup and zp are decomposable if and only if $z = z_{k+1k} \neq 0$; if this is so, $\operatorname{diag}(zup) = \operatorname{diag}(zp) = c \in X$,

$$\hat{c} = \begin{bmatrix} \dfrac{1}{z} & 0 \\ z & 0 \end{bmatrix}$$

and by (1.15)

$$\lambda(zup) = \lambda(zp) = \lambda(c) = \left(\frac{z}{|z|}\right)^{m_{k+1}-m_k} |z|^{s_{k+1}-s_k}.$$

Now let μ be the character of C defined by (1.13). Applying Lemma 2 we obtain

$$\gamma(m,s) = i^{|m|}\,\pi\,2^s \left(\frac{z}{|z|}\right)^{m_k-m_{k+1}} |z|^{-2+s_k-s_{k+1}}. \tag{3.5}$$

The operators $A(p,\lambda)$ are defined in $L_2(V)$ at least formally by the integral

$$A(p,\lambda)f(v) = \frac{1}{\gamma(\lambda)}\int_Z \lambda^{-1}(zp)\mu^{\frac{1}{2}}(zp)f(z^{-1}v)\,dz. \tag{3.4}$$

The factor $\gamma(\lambda) = \gamma(m_k - m_{k+1}, s_k - s_{k+1})$, where for any integer m and complex number s

$$\gamma(m, s) = i^{|m|}\pi\, 2^s \frac{\Gamma\left(\frac{|m| + s}{2}\right)}{\Gamma\left(\frac{|m| + 2 - s}{2}\right)}. \tag{3.5}$$

To study the operators $A(p, \lambda)$ we introduce another class of operators. If m is an integer and s is a complex number, $0 < Re(s) < 1$, we define an operator $A(p, m, s)$ in $L_2(V)$ by

$$A(p, m, s)f(v) = \frac{1}{\gamma(m, s)} \int \left(\frac{z}{|z|}\right)^m |z|^{-2+s} f(z^{-1}v)\, dz. \tag{3.6}$$

As we shall show, the integral in (3.6) is well defined and $A(p, m, s)f \in L_2(V)$ for a dense set of functions f in $L_2(V)$. Thus (3.4) will define an operator in $L_2(V)$ whenever $0 < Re(s_k - s_{k+1}) < 1$. For the domain D of the operators $A(p, m, s)$ we shall take all bounded Baire functions with compact support.

LEMMA 3. *Let m be an integer, s a complex number, $0 < Re(s) < 1$, and f an element of D. Then*

(1) (3.6) *converges absolutely for all $v \in V$;*

(2) *$A(p, m, s)f \in L_2(V)$;*

(3) *the mapping $s \to A(p, m, s)f$ has a unique extension to $Re(s) = 0$, so that as a function of s with values in $L_2(V)$ it is continuous in $0 \leqq Re(s) < 1$;*

(4) *if $Re(s) = 0$, the operators $A(p, m, s)$ are isometric and extend (uniquely) to unitary operators on $L_2(V)$;*

(5) *if $Re(s_1) = Re(s_2) = 0$ then*

$$A(p, m_1, s_1)A(p, m_2, s_2) = A(p, m_1 + m_2, s_1 + s_2)$$

for all integers m_1 and m_2;

(6) *$A(p, 0, 0) = I =$ the identity.*

Proof. (1) If f is a function on V, it follows from (2.1) that we may write $f(v) = f(zz') = f(z, z')$. In addition, if f is integrable (2.2) shows that

$$\int f(v)\, dv = \int\int f(z, z')\, dz\, dz'.$$

Thus if we identitfy z with its entry z_{k+1k} and use the fact that Z is isomorphic to the additive group of complex numbers, we may rewrite (3.6) as

$$A(p, m, s)f(z, z') = \frac{1}{\gamma(m, s)} \int \left(\frac{w}{|w|}\right)^m |w|^{-2+s} f(z - w, z')\, dw. \tag{3.7}$$

In this integral w is a single complex variable, and dw is the 2-dimensional element of area. If we take absolute values in (3.7) and disregard the multiplicative constant, we have

$$|A(p,m,s)f(z,z')| \leqq \int |w|^{-2+s_0} |f(z+w,z')|\,dw$$

where $s_0 = Re(s)$, $0 < s_0 < 1$. Therefore

$$|A(p,m,s)f(z,z')|$$
$$\leqq \int_{|w|\leqq 1} |w|^{-2+s_0} |f(z+w,z')|\,dw + \int_{|w|>1} |w|^{-2+s_0} |f(z+w,z')|\,dw.$$

We shall denote the first integral by $g_1(z,z')$ and the second by $g_2(z,z')$. Since $0 < s_0 < 1$, $|w|^{-2+s_0}$ is integrable over the unit disc and square integrable over the set where $|w| > 1$. Because $|f(w,z')|$ is bounded and has compact support as a function of w, it therefore follows that the integrals defining g_1, g_2 exist for all z, z'. This proves (1).

(2) Let K be a compact set containing the support of $f(z,z')$, K_1 the projection of K on Z, and K_2 the projection of K on Z'. Then K_1, K_2 are both compact and have finite measures $|K_1|$, $|K_2|$. Denoting lub $|f|$ by M and using Young's inequality, we obtain

$$\int |g_1(z,z')|^2\,dz \leqq |K_1| \Big(\int_{|w|\leqq 1} |w|^{-2+s_0}\,dw\Big)^2 < \infty$$

and

$$\int |g_2(z,z')|^2\,dz \leqq M^2 |K_1|^2 \cdot \int_{|w|>1} |w|^{-4+s_0}\,dw < \infty.$$

If $g(z,z') = A(p,m,s)f(z,z')$, it follows that

$$\int |g(z,z')|^2\,dz \leqq \text{Constant} \cdot M^2(|K_1| + |K_2|^2).$$

Note that $f(z,z') = 0$ if $z' \notin K_2$. Hence $g(z,z') = 0$ when $z' \notin K_2$. Therefore

$$\int\int |g(z,z')|^2\,dz dz' \leqq \text{Constant} \cdot M^2(|K_1| + |K_1|^2)|K_2| < \infty$$

which proves (2).

(3) Let t be a complex variable. Then the Fourier transform of $f(z,z')$ with respect to the z variable is the function $\hat{f}(t,z')$ given by

$$\hat{f}(t,z') = \tfrac{1}{2\pi} \int e^{-i\,Re(t\bar{z})} f(z,z')\,dz. \tag{3.8}$$

Let us put $\hat{H}_\epsilon(w) = \dfrac{1}{\gamma(m,s)}\, e^{-\epsilon|w|} \Big(\dfrac{w}{|w|}\Big)^m |w|^{-2+s}$, $\epsilon \geqq 0$. We need the

following fact (see Bochner [1], p. 39). If $t \neq 0$

$$\lim_{\epsilon \downarrow 0} \int \hat{H}_\epsilon(w) e^{-i\,Re(t\bar{w})} dw = \left(\frac{t}{|t|}\right)^m |t|^{-s}.$$

We shall also put $g_\epsilon(z,z') = \int H_\epsilon(w) f(z-w,z') dw$ for $\epsilon \geqq 0$. Then $A(p,m,s)f(z,z') = g_0(z,z')$, and the argument used in (1) and (2) shows that g_ϵ, $\epsilon > 0$, also belongs to $L_2(V)$. Applying the dominated convergence theorem, we see that $g_\epsilon(z,z') \to g_0(z,z')$ for each z, z'. For $\epsilon \geqq 0$, g_ϵ is dominated by the $L_2(V)$ function

$$h(z,z') = \int \frac{|w|^{-2+s_0}}{|\gamma(m,s)|} |f(z-w,z')|\, dw.$$

A second application of domianted convergence shows that $g_\epsilon \to g_0$ in $L_2(V)$. We now denote by $\hat{g}_\epsilon(t,z')$ the Fourier transform (3.8) of $g_\epsilon(z,z')$, $\epsilon \geqq 0$. By the Plancherel theorem, we see that $\hat{g}_\epsilon(t,z') \to \hat{g}_0(t,z')$ in the $L_2(t,z')$ norm. On the other hand,

$$\hat{g}_\epsilon(t,z') = 2\pi \cdot H_\epsilon(t) \hat{f}(t,z'), \quad \epsilon > 0.$$

Since $2\pi \cdot H_\epsilon(t) \to \left(\frac{t}{|t|}\right)^m |t|^{-s}$, $t \neq 0$, we find that

$$(A(p,m,s)f)^\wedge = \hat{g}_0(t,z') = \left(\frac{t}{|t|}\right)^m |t|^{-s} \hat{f}(t,z'). \tag{3.9}$$

It follows from the Plancherel theorem that we may use (3.9) to extend the definition of $A(p,m,s)f$ to all s in the strip $0 \leqq Re(s) < 1$. To prove the continuity of $A(p,m,s)f$ as a function of s with values in $L_2(V)$ it is then sufficient to show that

$$\int\int \left|\left(\frac{t}{|t|}\right)^m \hat{f}(t,z')\right|^2 \left|\,|t|^{-s} - |t|^{-s_1}\right|^2 dt\, dz' \to 0$$

as $s \to s_1$, $0 \leqq Re(s) < 1$, $0 \leqq Re(s_1) < 1$. We recall that f is bounded and vanishes outside the compact set K; hence, $\hat{f}(t,z')$ is bounded and vanishes when $z' \notin K_2$. Thus the above expression is bounded by

$$\text{Constant} \cdot \int_{K_2} \int_{|t| \leqq 1} \left|\,|t|^{-s} - |t|^{-s_1}\right|^2 dt\, dz'$$
$$+ \int_{K_2} \int_{|t| \geqq 1} \left|\,|t|^{-s} - |t|^{-s_1}\right|^2 |\hat{f}(t,z')|^2 dt\, dz'.$$

An easy application of the Lebesgue dominated convergence theorem shows that both integrals tend to 0 as $s \to s_1$.

It is obvious that the mapping $s \to A(p,m,s)f$ has at most one continuous

extension to the strip $0 \leqq Re(s) < 1$, and the proofs of (4), (5) and (6) follows immediately from (3.9) and Plancherel's theorem. This concludes the proof of Lemma 3.

4. The intertwining properties of the operators $A(p,\lambda)$. Let S be the sub-group of G which is generated by the unitary matrices $p = p_k$, $1 \leqq k \leqq n-1$. Then S is contained in the normalizer of C, and there is little difficulty in showing that S is finite. For the purposes of this paper it will be convenient to refer to S as the Weyl group.[5]

If λ is a character of C and $q \in S$, the mapping $c \to \lambda(q^{-1}cp)$, $c \in C$ is again a character. We denote this character by $q\lambda$. Let us now suppose $a \to T(a,\lambda)$ and $a \to T(a,\lambda_1)$ are two members of the principal series. Then by a fundamental theorem of Gelfand and Neumark [3] these representations are unitarily equivalent if and only if there is an element q of S such that $\lambda_1 = q\lambda$.

Their proof of this result is in part abstract, and in part quite concrete; however, it yields only the existence of the intertwining operators.[6] In this section we shall exhibit specific intertwining operators between the representations $a \to T(a,\lambda)$ and $a \to T(a,p\lambda)$, $p = p_k$, $1 \leqq k \leqq n-1$.

Let λ be the character of C defined by

$$\lambda(c) = \prod_{j=1}^{m} \left(\frac{c_j}{|c_j|}\right)^{m_j} |c_j|^{s_j} \tag{4.1}$$

where $m_1, m_2, \cdots, m_n$ are integers and $s_1, s_2, \cdots, s_n$ are complex numbers such that $0 \leqq Re(s_j) < \frac{1}{2}$, $1 \leqq j \leqq n$. If $p = p_k$, $1 \leqq k \leqq n-1$ we shall put

$$A(p,\lambda) = A(p, m_k - m_{k+1}, s_k - s_{k+1})$$

It follows from Lemma 2 and (3.4) that $A(p,\lambda)$ is independent of the representation of λ given by (4.1). One can also see this directly. For if we also have

$$\lambda(c) \prod_{j=1}^{n} \left(\frac{c_j}{|c_j|}\right)^{m'_j} |c_j|^{s'_j}$$

where $m'_1, m'_2, \cdots, m'_n$ are integers and $s'_1, s'_2, \cdots, s'_n$ are complex numbers then there is an integer l and a complex number w such that $m_j = m'_j + l$ and $s_j = s'_j + w$, $1 \leqq j \leqq n$. Hence

$$m_j - m_{j+1} = m'_j - m'_{j+1} \text{ and } s_j - s_{j+1} = s'_j - s'_{j+1}, \qquad 1 \leqq j \leqq n.$$

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[5] The Weyl group as it is usually defined may be identified with a quotient group of S.

[6] In recent note (Doklady 1960, vol. 131, pp. 496-499; also Soviet Mathematics 1960, vol. 1, no. 2, pp. 276-279) Gelfand and Graev have announced some results dealing with integrals of the type (3.4).

When $Re(s_k - s_{k+1}) = 0$, $A(p,\lambda)$ is unitary on $L_2(V)$ (Lemma 3) and otherwise $A(p,\lambda)$ is unbounded—with domain D consisting of all bounded Baire functions vanishing outside compact subsets of V.

We recall that every continuous unitary character of C has the form (4.1) where $s_1, s_2, \cdots, s_n$ are purely imaginary, i.e., $Re(s_j) = 0$, $1 \leqq j \leqq n$.

THEOREM 1. *Let G be the $n \times n$ complex unimodular group, λ a continuous unitary character of the diagonal sub-group, and $a \to T(a,\lambda)$ the corresponding member of the principal series. Then if $p = p_k$, $1 \leqq k \leqq n-1$.*

$$(4.2) \qquad A(p,\lambda)T(a,\lambda) = T(a,p\lambda)A(p,\lambda)$$

for all $a \in G$.

Roughly speaking, the main idea of the proof is to first establish (4.2) for a class of non-unitary characters. The necessity for this arises from the fact that the operators $A(p,\lambda)$ are not given explicitly by (3.4) when λ is unitary. The result is then obtained for unitary characters by a limiting argument. In the proof we shall need a series of lemmas.

If $p = p_k$, $1 \leqq k \leqq n-1$, let μ_p be the character of C defined by

$$(4.3) \qquad \mu_p(c) = |c_k|^2 \, |c_{k+1}|^{-2}.$$

LEMMA 4. *Let $1 \leqq k \leqq n-1$, $p = p_k$, and $Z = Z(k, k+1)$. Suppose $z \in Z$, and $a = cu$, $c \in C$, $u \in U$. Then*

(1) $d(z\tilde{a}) = \mu_p(za)dz$;

(2) $\mu_p(zu) = \mu(zu)$;

(3) $\mu_p(c) = \mu^{\frac{1}{2}}(c)\mu^{\frac{1}{2}}(p^{-1}cp)$.

Proof. (1) By definition, μ_p is the character μ_{kk+1} given by (2.3). Thus if $b = \dot{a}$, it follows from (2.4) that $d(z\tilde{b}) = \mu_p(zb)dz$. On the other hand, since $b = \dot{c}\dot{u}$, $z\tilde{a} = (z\tilde{c})\tilde{u} = z\tilde{b}$, $z \in Z$. Hence, $d(z\tilde{a}) = \mu_p(zb)dz$. We also have $\mu_p(za) = \mu_p(cc^{-1}zcu) = \mu_p(c)\mu_p(c^{-1}zcu)$. By (4.3), $\mu_p(c) = \mu_p(\dot{c})$, and Lemma 1 shows that $\mu_p(c^{-1}zcu) = \mu_p(c^{-1}zc\dot{u})$. Thus

$$\mu_p(za) = \mu_p(\dot{c})\mu_p(c^{-1}zc\dot{u}) = \mu_p(ab).$$

(2) Suppose zu is decomposable. Then by Lemma 1, $\operatorname{diag}(zu) = c$ belongs to $X(k, k+1)$. Hence $c_k c_{k+1} = 1$, and from (1.13) we find that $\mu(c) = |c_k|^{2(n-2k+1)} \, |c_{k+1}|^{2(n-2k-1)} = |c_k|^2 \, |c_{k+1}|^{-2} = \mu_p(c)$. Thus $\mu(zu) = \mu(c) = \mu_p(c) = \mu_p(zu)$.

(3) It is easily verified that $(p^{-1}cp)_k = c_{k+1}$, $(p^{-1}cp)_{k+1} = c_k$, and $(p^{-1}cp)_j = c_j$ for $j \neq k$ and $j \neq k+1$. Thus

$$\begin{aligned}\mu^{\frac{1}{2}}(c)\mu^{-\frac{1}{2}}(p^{-1}cp) &= |c_k|^{n-2k+1}|c_{k+1}|^{n-2k-1}|c_{k+1}|^{2k-n-1}|c_k|^{2k-n+1}\\ &= |c_k|^2|c_{k+1}|^{-2}\\ &= \mu_p(c).\end{aligned}$$

This establishes (3) and concludes the proof of Lemma 4.

Let a be a decomposable element of G and λ a continuous character of C. In order to simplify the notation we shall set

$$K(a,\lambda) = \lambda(a)\mu^{\frac{1}{2}}(a). \tag{4.4}$$

If a is not decomposable, we put $K(a,\lambda) = 0$. It follows that

$$K(va,\lambda) = \lambda(va)\mu^{\frac{1}{2}}(va), \quad v \in V, \ a \in G \tag{4.5}$$

and that

$$K(zp,\lambda^{-1}) = \lambda^{-1}(zp)\mu^{\frac{1}{2}}(zp), \quad z \in Z. \tag{4.6}$$

We recall that D is the collection of all bounded Baire functions vanishing outside compact subsets of V. For f in D, let

$$T(a,\lambda)f(v) = K(va,\lambda)f(v\tilde{a}). \tag{4.7}$$

It will also be convenient to denote the character contragredient to a given character λ and λ'. Thus

$$\lambda'(c) = \bar{\lambda}(c^{-1}), \quad c \in C. \tag{4.8}$$

In the next three lemmas we suppose the following:

(i) $p = p_k$, $1 \leqq k \leqq n-1$.

(ii) $a \in G$, $f \in D$, and $g \in D$.

(iii) λ is given by (4.1) and $0 < Re(s_k - s_{k+1}) < 1$.

(iv) $\lambda_1 = (p\lambda)'$.

In addition, we shall denote the inner product

$$\int f_1(v)\overline{f_2(v)}dv$$

of two functions f_1, f_2 in $L_2(V)$ by (f_1, f_2).

LEMMA 5. *Suppose $T(a,\lambda)f \in L_2(V)$. Then the integral*

$$\int_Z K(zva,\lambda)K(z^{-1}p,\lambda^{-1})f[(zv)\tilde{a}]dz$$

exists for almost all v in V, $A(p,\bar{\lambda})g \in L_2(V)$, and

$$(T(a,\lambda)f, A(p,\bar{\lambda})g)$$
$$= \frac{1}{\gamma(\lambda)} \int_V \overline{g(v)}\, dv \int_Z K(zva,\lambda) K(z^{-1}p, \lambda^{-1}) f[(zv)\tilde{a}]\, dz.$$

Proof. By Lemma 3, $A(p,\bar{\lambda})g \in L_2(V)$, and an easy computation shows that

$$\frac{K(z^{-1}p,\lambda^{-1})}{\gamma(\lambda)} = \frac{K(zp,\lambda')}{\gamma(\bar{\lambda})}. \tag{4.9}$$

Thus

$$(T(a,\lambda)f, A(p,\bar{\lambda})g)$$
$$= \frac{1}{\gamma(\lambda)} \int_V K(va,\lambda) f(v\tilde{a})\, dv \int_Z K(z^{-1}p,\lambda^{-1}) \bar{g}(z^{-1}v)\, dz.$$

Now, $K(va,\lambda)f(v\tilde{a})K(z^{-1}p,\lambda^{-1})\bar{g}(z^{-1}v)$ is a Baire function on $V \times Z$ and

$$\int_V |K(va,\lambda)f(v\tilde{a})|\, dv \int_Z |K(z^{-1}p,\lambda^{-1})\bar{g}(z^{-1}v)|\, dz < \infty. \tag{4.10}$$

Thus, by Fubini's theorem

$$(T(a,\lambda)f, A(p,\bar{\lambda})g) = \iint_{V\times Z} K(va,\lambda)K(z^{-1}p,\lambda^{-1})f(v\tilde{a})\bar{g}(z^{-1}v)\, dxdz.$$

Since the homeomorphism $(v,z) \to (zv,z)$ is measure preserving,

$$(T(a,\lambda)f, A(p,\bar{\lambda})g) = \iint_{V\times Z} K(zva,\lambda)K(z^{-1}p,\lambda^{-1})f[(zv)\tilde{a}]\bar{g}(v)\, dvdz.$$

The stated result now follows from a second application of Fubini's theorem.

LEMMA 6. *Suppose $T(a^{-1},\lambda_1)g \in L_2(V)$. Then the integral*

$$\int_Z K(va,p\lambda)K(z^{-1}p,\lambda^{-1})f[z(v\tilde{a})]\, dz$$

exists for almost all v in V, and

$$(A(p,\lambda)f, T(a^{-1},\lambda_1)g) = \int_V \bar{g}(v)\, dv \int_Z K(va,p\lambda)K(z^{-1}p,\lambda^{-1})f[z(v\tilde{a})]\, dz.$$

Proof. By Lemma 3, $A(p,\lambda)f \in L_2(V)$. Thus

$$(A(p,\lambda)f, T(a^{-1},\lambda_1)g)$$
$$= \frac{1}{\gamma(\lambda)} \int A(p,\lambda)f(v)K(va^{-1},\bar{\lambda}_1)\bar{g}(v\tilde{a}^{-1})\, dv.$$

On the other hand, since $d(v\tilde{a}) = \mu(va)dv$,

$$(A(p,\lambda)f, T(a^{-1},\lambda_1)g) = \frac{1}{\gamma(\lambda)}\int A(p,\lambda)f(v\tilde{a})K[(v\tilde{a})a^{-1},\bar{\lambda}_1]\bar{g}(v)\mu(va)dv.$$

By (1.17), $K[(v\tilde{a}^{-1},\bar{\lambda}_1] = \lambda_1'(va)\mu^{-\frac{1}{2}}(va)$, and as $\lambda_1' = p\lambda$ we find that

$$\begin{aligned}(A(p,\ )f, T(a^{-1},\lambda_1)g)&\\ &= \frac{1}{\gamma(\lambda)}\int A(p,\lambda)f(v\tilde{a})K(va,p\lambda)\bar{g}(v)dv\\ &= \frac{1}{\gamma(\lambda)}\int_V \bar{g}(v)dv\int_Z K(va,p\lambda)K(z^{-1}p,\lambda^{-1})f[z(v\tilde{a})]dz.\end{aligned}$$

LEMMA 7. *Suppose both $T(a,\lambda)f$ and $T(a^{-1},\lambda_1)g$ belong to $L_2(V)$. Then*

$$(T(a,\lambda)f, A(p,\bar{\lambda})g) = (A(p,\lambda)f, T(a^{-1},\lambda_1)g).$$

Proof. It is easily verified that

$$K(z^{-1}p,\lambda^{-1}) = (-1)^{m_k - m_{k+1}}K(zp,\lambda^{-1}). \tag{4.11}$$

By Lemmas 5 and 6, it is therefore sufficient to show that for almost all v

$$\int_Z K(zva,\lambda)K(ap,\lambda)f[(zv)\tilde{a}]dz = \int_Z K(va,p\lambda)K(zp,\lambda^{-1})f[z(v\tilde{a})]dz. \tag{4.12}$$

In order to do this, we first observe that corresponding to a, there is a measurable subset V_a of V such that (i) $V - V_a$ has measure 0, (ii) the integrals in (4.12) exist for $v \in V_a$, and (iii) va is decomposable for each $v \in V_a$. To each $v \in V_a$ there is a corresponding element $b = uc$ in UC such that $va = b(v\tilde{a})$. In addition, there is a measurable subset $Z_{v,a}$ of Z, depending on v and a, with the property that $Z - Z_{v,a}$ has measure 0 and ab is decomposable for each $z \in Z_{v,a}$. Since the mapping $z \to z\tilde{b}$ carries Z into Z and $d(z\tilde{b}) = \mu_p(zb)dz$ (Lemma 4), it follows that the second integral in (4.2) is equal to

$$\int_{Z_{v,a}} K(va,p\lambda)K[(z\tilde{b})p,\lambda^{-1}]f[(a\tilde{b})(v\tilde{a})]\mu_p(zb)dz. \tag{4.13}$$

If z is a fixed element of $Z_{v,a}$ there are matrices u_1 and c_1 in U and C respectively such that $zu = u_1c_1(z\tilde{u})$. Since $b = uc$, $zb = zuc = u_1c_1c \cdot c^{-1}(zu)c = u_1c_1c(z\tilde{b})$. Thus $\operatorname{diag}(zb) = c_1c$ and $zva = zb(v\tilde{a}) = u_1 \cdot c_1c \cdot (z\tilde{b})(v\tilde{a})$. It follows that zva is decomposable, that $\operatorname{diag}(zva) = c_1c$, and that $(zv)\tilde{a} = (z\tilde{b})(v\tilde{a})$. Hence upon comparing (4.13) with the first integral in (4.12) we see that it is sufficient to prove

$$K(zp,\lambda^{-1})K(c_1c,\lambda) = K(c,p\lambda)\mu_p(c_1c)K[(z\tilde{b})p,\lambda^{-1}]. \tag{4.14}$$

Now from the definition (4.4) of K and the multiplicative properties (1.16), (1.17) of the functions λ, μ it follows that

$$\begin{aligned} K[(z\bar{b})p,\lambda^{-1}] &= K[c^{-1}(z\tilde{u})cp,\lambda^{-1}] \\ &= K(c^{-1},\lambda^{-1})K[(z\tilde{u})p\cdot p^{-1}cp,\lambda^{-1}] \\ &= K(c^{-1},\lambda^{-1})K(p^{-1}cp,\lambda^{-1})K[(z\tilde{u})p,\lambda^{-1}]. \end{aligned}$$

By Lemma 1, $K(zp,\lambda^{-1}) = K(zup,\lambda^{-1})$, and on the other hand,

$$\begin{aligned} K(zup,\lambda^{-1}) &= K(zu,\lambda^{-1})K[(z\tilde{u})p,\lambda^{-1}] \\ &= K(c_1,\lambda^{-1})K[(z\tilde{u})p,\lambda^{-1}]. \end{aligned}$$

From this we obtain the relation

$$K[(z\bar{b})p,\lambda^{-1}] = K(c^{-1},\lambda^{-1})K(p^{-1}cp,\lambda^{-1})K(c_1^{-1},\lambda^{-1})K(zp,\lambda^{-1}).$$

Therefore (4.14) reduces to the condition

$$\begin{aligned} &K(c_1,\lambda)K(c_1,\lambda^{-1})\mu_p(c_1^{-1}) \\ &\quad = K(c^{-1},\lambda)K(c,p\lambda)\mu_p(c)K(c^{-1},\lambda^{-1})K(p^{-1}cp,\lambda^{-1}). \end{aligned} \tag{4.15}$$

The left side of (4.15) equals $\mu(c_1)\mu_p(c_1^{-1})$, and this equals 1 by part (2) of Lemma 4.

In the right side of (4.15), the product $K(c,p\lambda)K(p^{-1}cp,\lambda^{-1}) = \mu^{\frac{1}{2}}(c)\mu^{\frac{1}{2}}(p^{-1}cp)$, while $K(c^{-1},\lambda)K(c^{-1},\lambda^{-1}) = \mu^{\frac{1}{2}}(c^{-1})\mu^{\frac{1}{2}}(c^{-1})$. Thus the right side of (4.15) reduces to $\mu_p(c)\mu^{-\frac{1}{2}}(c)\mu^{\frac{1}{2}}(p^{-1}cp)$, and this equals 1 by part (3) of Lemma 4. This finishes the proof of (4.12).

LEMMA 8. *Let λ be a continuous unitary character of C and $p = p_k$, $1 \leqq k \leqq n-1$. For $0 < s < \frac{1}{2}$ and $c \in C$, let $\lambda^{(s)}(c) = |c_k|^s\lambda(c)$, and $\lambda_1^{(s)}(c) = |c_{k+1}|^{-s}\lambda(p^{-1}cp)$. Then, corresponding to each a in G, there exist subsets $D(a,\lambda)$ and $D(a^{-1},p\lambda)$ of D with the following properties.*

(1) *$D(a,\lambda)$ and $D(a^{-1},p\lambda)$ are dense in $L_2(V)$.*

(2) *For f in $D(a,\lambda)$*

$$\lim_{s\downarrow 0} \| T(a,\lambda)f - T(a,\lambda^{(s)})f \|_2 = 0.$$

(3) *For g in $D(a^{-1},p\lambda)$*

$$\lim_{s\downarrow 0} \| T(a^{-1},p\lambda)g - T(a^{-1},\lambda^{(s)})g \|_2 = 0.$$

Proof. Let $f \in D$. Then by (1.7) and the definition of $T(a,\lambda^{(s)})f$, it follows that

$$\begin{aligned} T(a,\lambda^{(s)})f(v) &= [h(v)]^s T(a,\lambda)f(v) \\ &= [h(v)]^s \mu(va)\mu^{\frac{1}{2}}(va)f(v\tilde{a}) \end{aligned}$$

where $h(v) = \left| \dfrac{D_k(va)}{D_{k+1}(va)} \right|$. We also have

$$T(a^{-1}, \lambda_1^{(s)})f(v) = [h_1(v)]^s T(a^{-1}, p\lambda)f(v)$$

where $h_1(v) = \left| \dfrac{D_{k+2}(va)}{D_{k+1}(va)} \right|$.

Let E be the collection of all functions f in $L_2(V)$ with the property that $h^{\frac{1}{2}} \cdot T(a, \lambda)f$ belongs to $L_2(V)$. Since h is measurable and $T(a, \lambda)$ is unitary, E is dense in $L_2(V)$. If f is in D and $\epsilon > 0$, standard measure theoretic arguments show the existence of a function f_ϵ in D such that $\| f - f_\epsilon \|_2 < \epsilon$ and $|f_\epsilon(v)| \leqq |f(v)|$ a.e.. Hence $|T(a, \lambda)f_\epsilon(v)| \leqq |T(a, \lambda)f(v)|$ a.e., and therefore f_ϵ belongs to E. Let $D(a, \lambda) = D \cap E$. What we have just done shows $D(a, \lambda)$ is dense in $L_2(V)$.

If $f \in D(a, \lambda)$ and $0 < s < \frac{1}{2}$ then

$$|h(v)|^s \, |T(a, \lambda)f(v)| \leqq (h^{\frac{1}{2}}(v) + 1) \, |T(a, \lambda)f(v)| \tag{4.16}$$

for all v in V. In addition,

$$T(a, \lambda)f(v) = \lim_{s \downarrow 0} |h(v)|^s \, T(a, \lambda)f(v).$$

Since the right side of (4.16) is a function in $L_2(V)$, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{s \downarrow 0} \| T(a, \lambda)f - T(a, \lambda^{(s)})f \|_2 = 0.$$

Now let $D(a^{-1}, p\lambda)$ be the collection of all functions f in D such that $h_1^{\frac{1}{2}} \cdot T(a^{-1}, p\lambda)f$ belongs to $L_2(V)$. Then the validity of (3) and the fact that $D(a^{-1}, p\lambda)$ is dense in $L_2(V)$ are seen by exactly the same arguments as we have used above.

Proof of Theorem 1. We suppose $p = p_k$, $1 \leqq k \leqq n - 1$, and that λ is a continuous unitary character of C. Let $\lambda^{(s)}$, $\lambda_1^{(s)}$, $D(a, \lambda)$, and $D(a^{-1}, p\lambda)$ be defined as in Lemma 8. Then in order to verify (4.2), it is sufficient to show that

$$(A(p, \lambda)T(a, \lambda)f, g) = (T(a, p\lambda)A(p, \lambda)f, g)$$

for $f \in D(a, \lambda)$ and $g \in D(a^{-1}, p\lambda)$. Now (5) and (6) of Lemma 3 tell us that the adjoint, $A(p, \lambda)^*$, of $A(p, \lambda)$ equals $A(p, \bar{\lambda})$. On the other hand, the adjoint of the unitary operator $T(a, p\lambda)$ equals $T(a^{-1}, p\lambda)$. Thus, to prove the theorem, we need only show that

$$(T(a, \lambda)f, A(p, \bar{\lambda})g) = (A(p, \lambda)f, T(a^{-1}, p\lambda)g) \tag{4.17}$$

for $f \in D(a, \lambda)$ and $g \in D(a^{-1}, p\lambda)$. Let $0 < s < \frac{1}{2}$, $f \in D(a, \lambda)$, and $g \in D(a^{-1}, p\lambda)$. Then both $T(a, \lambda^{(s)})f$ and $T(a^{-1}, \lambda_1^{(s)})g$ belong to $L_2(V)$. In addition, $\lambda_1^{(s)} = [p \cdot \lambda^{(s)}]$, i.e., $\lambda_1^{(s)}$ is the contragradient of $p \cdot \lambda^{(s)}$. It follows from Lemma 7 that

$$(4.18) \qquad (T(a, \lambda^{(s)})f, A(p, \lambda^{(s)})g) = (A(p, \overline{\lambda^{(s)}})f, T(a^{-1}, \lambda_1^{(s)})g).$$

As s tends to 0 through positive values, the left and right sides of (4.18) tend to the left and right sides respectively of (4.17) (Lemma 3 and Lemma 8). This concludes the proof of Theorem 1.

5. Commutation relations satisfied by the operators $A(p, \lambda)$. Let $p = p_k$ and $q = p_j$, $1 \leqq j, k \leqq n-1$. Then by (4.2), $A(q, \lambda)T(a, \lambda) = T(a, q\lambda)A(q, \lambda)$ for all unitary characters λ. It follows that

$$\begin{aligned} A(p, q\lambda)A(q, \lambda)T(a, \lambda) &= A(p, q\lambda)T(a, q\lambda)A(q, \lambda) \\ &= T(a, pq\lambda)A(p, q\lambda)A(q, \lambda). \end{aligned}$$

Here we have used the fact that $(pq)\lambda = p(q\lambda)$. Thus $A(p, q\lambda)A(q, \lambda)$ is an intertwining operator for the representations, $a \to T(a, \lambda)$ and $a \to T(a, pq\lambda)$. Let us now suppose that $1 \leqq k \leqq n-2$ and that $q = p_{k+1}$, $p = p_k$. It is easy to see that $pqp = qpq$. Thus we can write down two intertwining operators for the representations corresponding to λ and $pqp\lambda$, namely

$$A(p, pq\lambda)A(q, p\lambda)A(p, \lambda)$$

and $A(q, pq\lambda)A(p, q\lambda)A(q, \lambda)$. We shall show that these apparently distinct operators are in fact identical. That they differ by at most a scalar factor of absolute value 1 is clear from the fact [3] that the representations of the principal series are irreducible. Our proof will not use irreducibility. We present it because it brings to light additional properties of the operators that will be used later on.

The operator $A(p, \lambda)$ equals $A(p, m_k - m_{k+1}, s_k - s_{k+1})$ where λ is given by (4.1) and $Re(s_j) = 0$, $1 \leqq j \leqq n$. From the definition of q, it follows that $A(p, q\lambda) = A(p, m_k - m_{k+2}, s_k - s_{k+2})$. Similarly,

$$A(q, \lambda) = A(q, m_{k+1} - m_{k+2}, s_{k+1} - s_{k+2}) \text{ and}$$
$$A(q, p\lambda) = A(q, m_k - m_{k+2}, s_k - s_{k+2}).$$

We also have

$$A(p, qp\lambda) = A(p, m_{k+1} - m_{k+2}, s_{k+1} - s_{k+2}), \text{ and}$$
$$A(q, pq\lambda) = A(q, m_k - m_{k+1}, s_k - s_{k+1}).$$

Using these relations we see that the equality of the above intertwining operators for all λ is equivalent to the validity of

$$(5.1)\qquad \begin{aligned} &A(p,m_1,s_1)A(q,m_1+m_2,s_1+s_2)A(p,m_2,s_2)\\ &\qquad = A(q,m_2,s_2)A(p,m_1+m_2,s_1+s_2)A(q,m_1,s_1)\end{aligned}$$

for all integers m_1, m_2 and all complex numbers s_1, s_2—subject to the conditions, $Re(s_1)=0$, $Re(s_2)=0$. If we replace m_1, s_1 by $-m_1$, $-s_1$ and use (5) of Lemma 3 we find that (5.1) is equivalent to

$$(5.2)\qquad \begin{aligned} &A(p,-m_1,-s_1)A(q,-m_1,-s_1)A(q,m_2,s_2)A(p,m_2,s_2)\\ &\qquad = A(q,m_2,s_2)A(p,m_2,s_2)A(p,-m_1,-s_1)A(q,-m_1,-s_1).\end{aligned}$$

By (5) and (6) of Lemma 3

$$A(p,-m_1,-s_1)A(q,-m_1,-s_1)=[A(q,m_1,s_1)A(p,m_1,s_1)]^{-1}.$$

Thus (5.2) is equivalent to the statement that the unitary operators $A(q,m,s)A(p,m,s)$, $Re(s)=0$ form a commutative family;[7] however, the operators $A(q,m,s)$ and $A(p,m,s)$ do not commute (when $p=p_k$, $q=p_{k+1}$).

THEOREM 2. *Let* $p=p_k$ *and* $q=p_j$, $1\leqq k<j\leqq n-1$. *Then the operators* $A(q,m,s)A(p,m,s)$, $Re(s)=0$, *m an integer, form a commutative family of unitary operators on* $L_2(V)$, *and* $A(q,m,s)$ *commutes with* $A(p,m,s)$ *for all m, s if and only if* $j-k>1$.

Proof. Suppose $j-k>1$. Then it is easily seen that the matrices in $Z(k,k+1)$ commute with those in $Z(j,j+1)$ (see Section 2). From this observation and (3.6) it follows by a simple argument that $A(p,m,s)$ commutes with $A(q,m,s)$.

We suppose now that $j=k+1$ and consider the decomposition $V=ZZ'$ where $Z=Z(k,k+2)$. Then Z is isomorphic to $V(3)$ and contains the subgroups $Z(k,k+1)$ and $Z(k+1,k+2)$. In addition $dv=dzdz'$, and if f is a function on V, we may write $f(v)=f(zz')=f(z,z')$. If f is a bounded Baire function with compact support on V, and $0<Re(s)<1$, it therefore follows from (3.6) that

$$(5.3a)\qquad A(q,m,s)f(z,z')=\frac{1}{\gamma(m,s)}\int\left(\frac{z_k}{|z_k|}\right)^m|z_k|^{-2+s}f(z_k^{-1}z,z')\,dz_k$$

and

$$(5.3b)\qquad A(p,m,s)f(z,z')=\frac{1}{\gamma(m,s)}\int\left(\frac{z_{k+1}}{|z_{k+1}|}\right)^m|z_{k+1}|^{-2+s}f(z_{k+1}^{-1}z,z')\,dz_{k+1}.$$

[7] The various equivalent forms of (5.1) were pointed out to us by W. F. Stinespring.

In (5.3a) and (5.3b) the integration is extended over the sub-groups $Z(k,k+1)$ and $Z(k+1,k+2)$ respectively. Thus neither of our operators affects the second variable z', and it is therefore sufficient to prove the remainder of the theorem in the case $n=3$. Thus in what follows we shall consider the group $V(3)$ consisting of all complex matrices of the form

$$v=\begin{bmatrix}1 & 0 & 0\\ v_{21} & 1 & 0\\ v_{21} & v_{32} & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ v_1 & 1 & 0\\ v_3 & v_2 & 1\end{bmatrix}.$$

We shall sometimes denote an element v of $V(3)$ by (v_1,v_2,v_3). If $w=(w_1,w_2,w_3)$ then

$$wv=(w_1+v_1,w_2+v_2,w_3+v_3+w_2v_1) \tag{5.4}$$

and $w^{-1}=(-w_1,-w_2,w_1w_2-w_3)$. The sub-groups $Z(1,2)$ and $Z(2,3)$ are given as follows: $Z(1,2)$ consists of all elements of the form $z_1=(z_1,0,0)$ while $Z(2,3)$ is the set of all elements of the form $(0,z_2,0)$. Rewriting (5.3a) and (5.3b) with the aid of (5.4) we obtain

$$\begin{aligned}A(p,m,s)f(v)&=\frac{1}{\gamma(m,s)}\int\left(\frac{z_1}{|z_1|}\right)^m|z_1|^{-2+s}f(z_1^{-1}v)\,dz_1\\&=\frac{1}{\gamma(m,s)}\int\left(\frac{z}{|z|}\right)^m|z|^{-2+s}f(v_1-z,v_2,v_3)\,dz\end{aligned} \tag{5.5a}$$

and

$$\begin{aligned}A(q,m,s)f(v)&=\frac{1}{\gamma(m,s)}\int\left(\frac{z_2}{|z_2|}\right)^m|z_2|^{-2+s}f(z_2^{-1}v)\,dz_2\\&=\frac{1}{\gamma(m,s)}\int\left(\frac{z}{|z|}\right)^m|z|^{-2+s}f(v_1,v_2-z,v_3-zv_1)\,dz.\end{aligned} \tag{5.5b}$$

When $Re(s)=0$, $A(p,m,s)$ and $A(q,m,s)$ are defined as in Lemma 3.

To analyze the products $A(q,m,s)A(p,m,s)$, $Re(s)=0$, it will be convenient to introduce another unitary operator E, which is closely related to the Fourier-Plancherel transform on $V(3)$—studied by Godement in [4]. In defining E we first denote by E_1, the unitary operator characterized by the condition

$$E_1f(v_1,v_2,v_3=f(v_1-\frac{\bar{v}_2}{\bar{v}_3},v_2,v_3),\quad v_3\neq 0. \tag{5.6}$$

It follows easily from the classical Plancherel theorem that there is a unique unitary E_2 on $L_2(V)$ such that

$$(5.7)\qquad E_2 f(v_1, v_2, v_3) = \frac{1}{(2\pi)^2}\int e^{-iRe(v_2\bar{w}_2+v_3\bar{w}_3)} f(v_1, w_2, w_3)\,dw_2 dw_3$$

whenever $f \in L_2(V)$ and $f(v_1, w_2, w_3)$ is integrable with respect to w_2, w_3. We now put $E = E_1 E_2$.

LEMMA 9. *Let $Re(s) = 0$ and $f \in L_2(V)$. Then*

$$(5.8)\qquad EA(q, m, s) f(v_1, v_2, v_3) = \left(\frac{v_3\bar{v}_1}{|v_3\bar{v}_1|}\right)^m |v_3 v_1|^{-s} Ef(v_1, v_2, v_3).$$

Proof. Consider first the case in which $0 < Re(s) < 1$ and f is a bounded function with compact support. Then by Lemma 3, $A(q, m, s) f \in L_2(V)$. Let us put

$$H_\epsilon(z) = \frac{e^{-\epsilon|z|}}{\gamma(m, s)}\left(\frac{z}{|z|}\right)^m |z|^{-2+s},\quad \epsilon \geqq 0$$

and

$$g_\epsilon(v_1, v_2, v_3) = \int H_\epsilon(z) f(v_1, v_2 - z, v_3 - z v_1)\,dz.$$

Then $g_0 = A(p, m, s) f$, and $g_\epsilon \to g_0$ in $L_2(V)$ as $\epsilon \to 0$. The argument showing this is similar to one given in the proof of Lemma 3. If $\epsilon > 0$ we have

$$\begin{aligned}
\int\int |g_\epsilon(v_1, v_2, v_3)|\,dv_2 dv_3
&\leqq \int dv_2 dv_3 \int |H_\epsilon(z)|\,|f(v_1, v_2 - z, v_3 - z v_1|\,dz \\
&= \int |H_\epsilon(z)|\,dz \int |f(v_1, v_2 - z, v_3 - z v_1)|\,dv_2 dv_3 \\
&= \int |H_\epsilon(z)|\,dz \cdot \int |f(_1, v_2, v_3)|\,dv_2 dv_3 \\
&< +\infty.
\end{aligned}$$

Hence we may use (5.7) to compute $E_2 g_\epsilon$. Interchanging the order of integration and changing variables in the resulting triple integral, we find that

$$E_2 g_\epsilon(v) = \hat{f}(v_1, v_2, v_3) \int H_\epsilon(z) e^{-iRe[(v_2+v_3\bar{v}_1)\bar{z}]}\,dz.$$

Thus by (5.6)

$$Eg_\epsilon(v) = Ef(v) \int H_\epsilon(z) e^{-iRe(v_3\bar{v}_1\bar{z})}\,dz.$$

As $\epsilon \to 0$, $\| EA(q, m, s) f - Eg_\epsilon \|_2 \to 0$ and

$$\int H_\epsilon(z) e^{-iRe(v_3\bar{v}_1\bar{z})}\,dz \to \left(\frac{v_3\bar{v}_1}{|v_2\bar{v}_1|}\right)^m |v_3\bar{v}_1|^{-s},\quad v_3\bar{v}_1 \neq 0.$$

It should be noted that (5.8) holds for all f in $L_2(V)$ whenever $Re(s)=0$. It follows that

$$EA(q,m,s)f(v_1,v_2,v_3)=\left(\frac{v_3\bar{v}_1}{|v_3\bar{v}_1|}\right)^m |v_3v_1|^{-s}Ef(v_1,v_2,v_3).$$

We now let $s\to s_0$ where $Re(s_0)=0$. Since E is bounded, Lemma 3 shows that $EA(q,m,s)f\to EA(q,m,s_0)f$ in $L_2(V)$. On the other hand, the right side of the above equation tends to $\left(\frac{v_3\bar{v}_1}{|v_3\bar{v}_1|}\right)^m |v_3v_1|^{-s_0}Ef(v_1,v_2,v_3)$, both pointwise and in the norm. This proves (5.8) when f is bounded and has compact support. The validity of (5.8), for all functions in $L_2(V)$, now follows by routine arguments.

LEMMA 10. *If $Re(s)=0$, $A(p,m,s)$ commutes with E.*

Proof. Let $f(v_1,v_2,v_3)=g(v_1)h(v_2,v_3)$ where g and h are bounded and have compact supports. Then from (5.5a) and Lemma 3 we see that $A(p,m,s)f=(A(m,s)g)h$, $0\leqq Re(s)<1$, where $A(m,s)$ is the operator $A(p,m,s)$ in the case $n=2$; thus we have

$$A(m,s)g(v_1)=\frac{1}{\gamma(m,s)}\int\left(\frac{z}{|z|}\right)^m |z|^{-2+s}g(v_1-z)dz,\quad 0<Re(s)<1. \tag{5.9}$$

It now follows easily from (5.7) and (5.6) that $EA(p,m,s)f=A(p,m,s)Ef$, $0\leqq Re(s)<1$. Since $A(p,m,s)$ is bounded, for $Re(s)=0$ and finite sums of functions of the form gh are dense in $L_2(V)$, it follows that $A(p,m,s)$ commutes with E whenever $Re(s)=0$.

We have already defined the operators $A(m,s)$ on $L_2(v_1)$. It will be convenient at this point to introduce another family of operators $B(m,s)$. These are defined on $L_2(v_1)$, when m is an integer and $Re(s)=0$, by

$$B(m,s)f(v_1)=\left(\frac{v_1}{|v_1|}\right)^{-m}|v_1|^{-s}f(v_1). \tag{5.10}$$

It should be noted that $B(0,0)$ is the identity operator and that

$$B(m_1,s_1)B(m_2,s_2)=B(m_1+m_2,s_1+s_2).$$

Now let $Re(s)=0$ and suppose $f\in L_2(V)$. Then upon replacing f by $E^{-1}A(p,m,s)f$ in (5.8) and using Lemma 10, we obtain the relation

$$\begin{aligned}&EA(q,m,s)A(p,m,s)E^{-1}f(v_1,v_2,v_3)\\ &=\left(\frac{v_3}{|v_3|}\right)^m|v_3|^{-s}\left(\frac{v_1}{|v_1|}\right)^{-m}|v\ |^{-s}A(p,m,s)f(v_1,v_2,v_3).\end{aligned} \tag{5.11}$$

In (5.11) we next set $f = gh$ where $g \in L_2(v_1)$ and $h \in L_2(v_2, v_3)$. Rewriting the right side of (5.11), with the aid of (5.10), we find that

$$\begin{aligned} &EA(q,m,s)A(p,m,s)E^{-1}f(v_1,v_2,v_3) \\ (5.12)\qquad &= \left(\frac{v_3}{|v_3|}\right)^m |v_3|^{-s}(B(m,s)A(m,s)g(v_1))h(v_2,v_3). \end{aligned}$$

Thus the operators $A(q,m,s)A(p,m,s)$ form a commutative family if and only if the operators $B(m,s)A(m,s)$ do. We shall finish the proof of Theorem 2 by showing that this is indeed the case. We should also remark that $B(m,s)$ does not commute with $A(m,s)$ if $m \neq 0$ or $s \neq 0$.

LEMMA 11. *Let $V = V(2)$. Then the unitary operators $B(m,s)A(m,s)$ on $L_2(V)$, m an integer, and $Re(s) = 0$, form a commutative family.*

Proof. For each complex number $z \neq 0$, let L_z be the operator defined on Baire functions f by

$$L_z f(v) = f(z^{-1}v), \quad v \in V.$$

Then $L_z^{-1} = L_{z^{-1}}$ and since $\int |f(v)|^2 dv = |z|^{-2} \int |f(z^{-1}v)|^2 dv$, L_z is bounded on $L_2(dv)$; on the other hand, L_z is easily seen to be unitary on the space $L_2(|v|^{-2}dv)$. A straightforward computation shows that

$$(5.13)\qquad L_z B(m,s) L_z^{-1} = \left(\frac{z}{|z|}\right)^m |z|^s B(m,s).$$

Next consider $A(m,s)f$ where $0 < Re(s) < 1$ and f is a bounded function with compact support. A simple argument by change of variables shows

$$L_z A(m,s) L_z^{-1} f = \left(\frac{z}{|z|}\right)^{-m} |z|^{-s} A(m,s) f.$$

Using statement (3) of Lemma 3, we then obtain the relation

$$(5.15)\qquad L_z A(m,s) L_z^{-1} = \left(\frac{z}{|z|}\right)^{-m} |z|^{-s} A(m,s), \; Re(s) = 0.$$

Combining (5.13) and (5.15) we find that

$$(5.16)\qquad L_z B(m,s) A(m,s) = B(m,s) A(m,s) L_z, \quad z \neq 0.$$

In order to finish the proof, it will suffice, in view of (5.16), to show that the operators L_z, $z \neq 0$ generate a maximal abelian (self-adjoint) algebra of bounded operators on $L_2(dv)$. We shall prove this with the aid of the isometric mapping, M of $L_2(dv)$ onto $L_2(|v|^{-2}\,dv)$ which takes $f(v)$ into $vf(v)$. If $z \neq 0$, we have $L_z Mf(v) = Mf(z^{-1}v) = z^{-1}vf(z^{-1}v) = z^{-1}ML_z f(v)$. Thus

$L_zM = z^{-1}ML_z$ or $ML_zM^{-1} = zL_z$. Hence the weakly closed algebra generated by the operators ML_zM^{-1} on $L_2(|v|^{-2}dv)$ coincides with the algebra generated by the unitary operators L_z; since $|v|^{-2}dv$ is a Haar measure on the multiplicative group of complex numbers, it follows, by a well known result, that the translations L_z generate a maximal abelian algebra on $L_2(|v|^{-2}dv)$.

This concludes the proof of the lemma and the proof of Theorem 2 as well.

COROLLARY. *If m is an integer and $Re(s) = 0$, let $J(k, m, s)$, $1 \leq k \leq n-1$, be the unitary operator on $L_2(V)$ given by*

$$(5.17) \qquad J(k, m, s) = A(p_{n-1}, m, s) \cdots A(p_{k+1}, m, s) A(p_k, m, s).$$

Then for fixed k, the operators $J(k, m, s)$ obtained by varying m and s form a commutative family.

Proof. Since $J(n-1, m, s) = A(p_{n-1}, m, s)$, the corollary in this case follows from Lemma 3. If $n > 2$, $J(n-2, m, s) = A(p_{n-1}, m, s) A(p_{n-2}, m, s)$. This case is covered by the theorem. We may therefore suppose $n > 3$ and assume by induction on $n-k$ that the corollary is true for $2 \leq k \leq n-1$. Let us put $X_k = J(k, m_1, s_1)$ and $Y_k = J(k, m_2, s_2)$. We note that X_k commutes with Y_k if and only if $X_k Y_k^{-1} = Y_k^{-1} X_k$. Now

$$X_1 = X_3 A(p_2, m_1, s_1) A(p_1, m_1, s_1)$$

and $Y_1 = Y_3 A(p_2, m_2, s_2) A(p_1, m_1, s_1)$. Setting $B_{jk} = A(p_j, m_k, s_k)$, $1 \leq j$, $k \leq 2$, we find that

$$X_1 Y_1^{-1} = X_3 B_{21} B_{11} B_{12}^{-1} B_{22}^{-1} Y_3^{-1}.$$

Now by Theorem 3, $B_{21} B_{11} B_{12}^{-1} B_{22}^{-1} = B_{12}^{-1} B_{22}^{-1} B_{21} B_{11}$ and $B_{12}^{-1} X_3 = X_3 B_{12}^{-1}$; moreover, $B_{11} Y_1^{-1} = Y_1^{-1} B_{11}$. Thus $X_1 Y_1^{-1} = B_{12}^{-1} X_3 B_{22}^{-1} B_{21} Y_3^{-1} B_{11}$, and and since B_{22}^{-1} commutes with B_{21},

$$X_1 Y_1^{-1} = B_{12}^{-1} X_2 Y_2^{-1} B_{11}.$$

By induction, X_2 commutes with Y_2^{-1}; hence $X_1 Y_1^{-1} = Y_1^{-1} X_1$.

6. The normalized principal series. This section is devoted to the construction and study of a series of unitary representations of G which we shall call the normalized principal series. Before describing these representations it will be convenient to make several notational conventions.

Let λ be a continuous unitary character of the diagonal sub-group of G. Then we can write

$$(6.1) \qquad \lambda(c) = \prod_{j=1}^{n} \left(\frac{c_j}{|c_j|} \right)^{m_j} |c_j|^{s_j}, \quad c \in C$$

where $m_1, m_2, \cdots, m_n$ are integers and $s_1, s_2, \cdots, s_n$ are complex numbers such that $Re(s_j) = 0$, $1 \leqq j \leqq n$. Moreover, we can choose m_j and s_j so that

$$(6.2) \qquad 0 \leqq \sum_{j=1}^{n} m_j < n$$

and

$$(6.3) \qquad \sum_{j=1}^{n} s_j = 0.$$

Furthermore, this can be done in only one way; that is, (6.1), (6.2), (6.3) determine m_j and s_j uniquely. In what follows, we shall always assume these conditions are satisfied. Therefore, we may write $\lambda = (m_1, \cdots, m_n; s_1, \cdots, s_n)$. Let us also put $m_1 + \cdots + m_n = r$ and

$$(6.4) \qquad \operatorname{res} \lambda = (0, \cdots, r; 0, \cdots, 0).$$

We shall call the character $\operatorname{res} \lambda$ the residue of λ. The characters therefore split up into n distinct classes according to their residues.

Let p_k, $1 \leqq k \leqq n-1$, be the unitary matrix given by (3.1). We recall that $p_1, p_2, \cdots, p_{n-1}$ generate a finite group S which we have referred to as the Weyl group. If we regard the elements of G as acting by right multiplication on complex n-tuple space then p_k sends $(w_1, \cdots, w_k, w_{k+1}, \cdots, w_n)$ into $(w_1, \cdots, -w_{k+1}, w_k, \cdots, w_n)$. It follows that S is essentially the symmetric group of degree n. If

$$\lambda = (m_1, \cdots, m_k, m_{k+1}, \cdots, m_n; s_1, \cdots, s_k, s_{k+1}, \cdots, s_n)$$

then it is easily verified that

$$p_k \lambda = (m_1, \cdots, m_{k+1}, m_k, \cdots, m_n; s_1, \cdots, s_{k+1}, s_k, \cdots, s_n)$$

where $p_k \lambda(c) = \lambda(p_k^{-1} c p_k)$, $c \in C$. Hence if q is any element of S it follows that λ and $q\lambda$ have the same residue.

If $\lambda = (m_1, \cdots, m_n; s_1, \cdots, s_n)$ denote by $J(k, \lambda)$ the operator $J(k, m_k, s_k)$ defined by (5.17), and set

$$(6.5) \qquad W(\lambda) = J(1, \lambda) J(2, \lambda) \cdots J(n-1, \lambda).$$

Furthermore, if $a \rightarrow T(a, \lambda)$ is the representation of the principal series corresponding to λ, let

$$(6.6) \qquad R(a, \lambda) = W(\lambda) T(a, \lambda) W(\lambda)^{-1}, \quad a \in G.$$

Since $W(\lambda)$ is a unitary operator, (6.6) defines a continuous unitary representation $a \to R(a, \lambda)$ of G on $L_2(V)$. These new representations $a \to R(a, \lambda)$ are, by construction, unitarily equivalent to the corresponding representations $a \to T(a, \lambda)$ of the principal series. We shall call this new series of representations the *normalized principal series.*

The following theorem which describes the behaviour of $R(a, \lambda)$ as a function of λ is one of the main results of this paper.

THEOREM 3. *Let G be the complex unimodular group of degree n and Λ the collection of continuous unitary characters of the diagonal sub-group C. Denote by G_0 the sub-group of G which consists of all those elements $a \in G$ with the property that $a_{jn} = 0$, $1 \leqq j \leqq n-1$. If $a \in G$ and $\lambda \in \Lambda$, let $R(a, \lambda)$ be the operator of the normalized principal series defined by* (6.6).

(1) *Then for each fixed $a \in G_0$, $R(a, \lambda)$ as a function of λ depends only upon the residue of λ; in fact*

$$R(a, \lambda) = R(a, \operatorname{res} \lambda) = T(a, \operatorname{res} \lambda) \tag{6.7}$$

where $T(a, \operatorname{res}\lambda)$ is the operator (1.18) *of the principal series, corresponding to the element $a \in G_0$ and to the character* $\operatorname{res}\lambda$, *given by* (6.4).

(2) *In addition,*

$$R(a, \lambda) = R(a, q\lambda) \tag{6.8}$$

for all $a \in G$, $\lambda \in \Lambda$, and q in the Weyl group S.

Remarks. (i) If we allow the elements of G to act as before on complex n-tuple space, then G_0 is the largest sub-group of G leaving invariant the $n-1$ dimensional subspace of vectors $(w_1, \cdots, w_n)$ for which $w_n = 0$. Moreover, in the proof we shall show that a given element of G is either in G_0 or in $G_0 p_{n-1} G_0$. The study of the dependence of $R(a, \lambda)$ on λ for a general element $a \in G$ is therefore reduced by (1) to the study of the operators $R(p_{n-1}, \lambda)$. In particular, the problem of the analytic continuation of the operators $R(a, \lambda)$ to non-unitary characters is essentially reduced to the problem of the continuation of the operators $R(p_{n-1}, \lambda)$.

(ii) Given part (1) of the theorem, part (2) follows from the result of Gelfand and Neumark [3] that the representations of the principal series are irreducible when restricted to G_0. For the representations of the normalized series are then also irreducible on G, and moreover, the represen-

tations corersponding to λ and $q\lambda$ are unitarily equivalent. Thus there is a unitary operator $B(\lambda)$ such that

$$B(\lambda)R(a,\lambda)=R(a,q\lambda)B(\lambda),\ a\in G.$$

But since λ and $q\lambda$ have the same residue, $R(a,\lambda)=R(a,q\lambda)$, $a\in G_0$. Hence

$$B(\lambda)R(a,\lambda)=R(a,\lambda)B(\lambda),\ a\in G_0.$$

Because the representations are irreducible on G_0 it follows that $B(\lambda)$ is a non-zero scalar multiple of the identity. Thus $R(a,\lambda)=R(a,q\lambda)$ for all $a\in G$.

On the other hand, it is not difficult to prove (2) directly, and we shall give a proof which does not use the irreducibility of the representations of G_0.

LEMMA 12. *Let S be the group generated by $p_1,p_2,\cdots,p_{n-1}$ and H the lower triangular sub-group,* i. e., *the set of all $a\in G$ such that $a_{jk}=0$ for $j<k$. Then G is a finite union of double cosets of the form HqH where q is an element of S.*

Proof. See Gelfand and Neumark [3].

LEMMA 13. *G_0 is generated (algebraically) by H and $p_1,p_2,\cdots,p_{n-2}$.*

Proof. Let G_0' be the sub-group of G_0 consisting of all elements a such that $a_{nn}=1$ and $a_{nj}=0$, $1\leqq j\leqq n-1$. Then G_0' is isomorphic to $G(n-1)$ and contains $p_1,\cdots,p_{n-2}$. Moreover, it is easily seen that every element of G_0 can be expressed as the product of an element of G_0' with one in H. The lemma follows easily from these observations and the result of Lemma 12 applied to $G(n-1)$.

LEMMA 14. $G=G_0\cup G_0p_{n-1}G_0$.

Proof. Let $a\in G$. Then by Lemma 12 there is an element $q\in S$ such that $a\in HqH$. Let us suppose that a is not a member of G_0. Then Lemma 13 shows that q is not a member of the sub-group S' generated by $p_1,\cdots,p_{n-2}$. It follows that there exist elements q', q'' in S' such that $q=q'p_{n-1}q''$ and hence that $a\in G_0p_{n-1}G_0$. This is most easily seen from the interpretation of of S as the 'symmetric group' together with the following fact about permutations: If π is a permutation of the integers $1,2,\cdots,n$ which does not fix n then there exist permutations π', π'' fixing n such that $\pi=\pi'(n-1,n)\pi''$ where $(n-1,n)$ denotes the transposition interchanging n and $n-1$.

If w is a non-zero complex number we shall find it convenient, in what follows, to adopt the notational device of writing $[w]$ for $\dfrac{w}{|w|}$.

LEMMA 15. *Let m be an integer, $Re(s)=0$, and $1\leqq k\leqq n-1$. Then for each $c\in C$ and $\lambda\in\Lambda$*

$$(6.9)\qquad A(p_k,m,s)T(c,\lambda)=\left[\frac{c_k}{c_{k+1}}\right]^{-m}\left|\frac{c_k}{c_{k+1}}\right|^{-s}T(c,\lambda)A(p_k,m,s).$$

Proof. Let ϵ denote the identity character. Then since $\lambda(vc)=\lambda(c)$ for each λ in Λ and v in V it follows that $T(c,\lambda)=\lambda(c)T(c,\epsilon)$. Hence $\lambda(c)T(c,\lambda_1)=\lambda_1(c)T(c,\lambda)$ for all λ,λ_1 in Λ. Using this relation and (4.2) we find that

$$(6.10)\qquad A(p_k,\lambda_1)T(c,\lambda)=\frac{\lambda_1(p_k^{-1}cp_k)}{\lambda_1(c)}\cdot T(c,\lambda)A(p_k,\lambda_1).$$

If $\lambda_1=(m_1,\cdots,m_n;s_1,\cdots,s_n)$ an easy computation shows

$$\frac{\lambda_1(p_k^{-1}cp_k)}{\lambda_1(c)}=\left[\frac{c_k}{c_{k+1}}\right]^{m_{k+1}-m_k}\left|\frac{c_k}{c_{k+1}}\right|^{s_{k+1}-s_k}.$$

In addition, $A(p_k,\lambda_1)=A(p_k,m_k-m_{k+1},s_k-s_{k+1})$. Now choosing λ_1 so that $m=m_k-m_{k+1}$ and $s=s_k-s_{k+1}$ and substituting into (6.10) we obtain (6.9).

LEMMA 16. *Let $1\leqq j,k\leqq n-1$, $q=p_j$, and $p=p_k$. If m is an integer and $Re(s)=0$, let $M(q,m,s)$ be the operator on $L_2(V)$ given by*

$$(6.11)\qquad M(q,m,s)f(v)=[v_{j+1\,j}]^{-m}\,|v_{j+1\,j}|^{-s}f(v).$$

In addition, let

$$(6.12)\qquad T(q,m,s)=M(q,m,s)T(q,\epsilon)$$

where $T(q,\epsilon)$ is the operator of the principal series corresponding to q,ϵ. Then

$$(6.13)\qquad A(p,m,s)T(p,m,s)=T(p,-m,-s)A(p,m,s);$$

$$(6.14)\qquad A(p,m,s)T(q,m',s')=T(q,m',s')A(p,m,s)\ \text{if}\ |k-j|>1;$$

$$(6.15)\qquad A(p,m,s)T(q,m',s')=T(q,m+m',s+s')A(p,m,s)\ \text{if}\ |k-j|=1.$$

Proof. Let $\lambda=(m_1,\cdots,m_n;s_1,\cdots,s_n)$. Then (1.18) and Lemma 2 show that

$$(6.16)\qquad T(q,\lambda)=T(q,m_j-m_{j+1},s_j-s_{j+1}).$$

We also have the relation

$$A(p,\lambda)=A(p,m_k-m_{k+1},s_k-s_{k+1}).$$

Furthermore,

$$T(p, p\lambda) = T(p, m_{j+1} - m_j, s_{j+1} - s_j).$$

Putting these observations together and using (4.2) we obtain (6.13).

If $|k - j| > 1$ then $T(q, \lambda) = T(q, p\lambda)$ and (6.14) follows immediately from (4.2).

In proving (6.15) we shall suppose $j = k + 1$, the argument being entirely similar in the case $k = j + 1$. If m, s, m', and s' are given we can always choose λ so that $m = m_k - m_{k+1}$, $s = s_k - s_{k+1}$, $m' = m_{k+1} - m_{k+2}$, and $s' = s_{k+1} - s_{k+2}$. Then it is also true that $m + m' = m_k - m_{k+2}$ and $s + s' = s_k - s_{k+2}$. Moreover,

$$T(q, p\lambda) = T(q, m_k - m_{k+2}, s_k - s_{k+2}).$$

From these relations and (4.2) we obtain (6.15).

Proof of Theorem 3. (1) Since $R(a_1a_2, \lambda) = R(a_1, \lambda)R(a_2, \lambda)$ for all a_1, a_2 in G and $\lambda \in \Lambda$, the validity of (6.7) for a_1, a_2 in G_0 implies its validity for their product a_1a_2. Now $H = VC$, and by Lemma 13, G_0 is generated by $p_1, \cdots, p_{n-2}$ and H. Thus it suffices to prove (6.7) when $a \in V$, $a \in C$, and $a = p_j$, $1 \leqq j \leqq n - 2$.

Let us suppose first that $a \in V$. Then in (1.18) $\lambda(va)\mu^{\frac{1}{2}}(va) = 1$ for all $v \in V$. Thus $T(a, \lambda_1) = T(a, \lambda_2)$ for all λ_1, λ_2 in Λ. Using this relation and (4.2) we find that the operators $A(p_k, m, s)$ commute with $T(a, \lambda)$. The operators $J(k, m, s)$ given by (5.17) therefore commute with $T(a, \lambda)$ and hence by (6.5) we see that $W(\lambda)$ also commutes with $T(a, \lambda)$. Thus $R(a, \lambda) = T(a, \lambda)$ and we have established (6.7) in the case that $a \in V$.

We now assume that $a = c \in C$. We also suppose $\lambda = (m_1, \cdots, m_n; s_1, \cdots, s_n)$ and that conditions (6.2) and (6.3) are satisfied. Then rewriting (6.1) we find that

$$\lambda(c) = \prod_{k=1}^{n-1} \left[\frac{c_k}{c_{k+1}}\right]^{\alpha_k} \left|\frac{c_k}{c_{k+1}}\right|^{\beta_k} \operatorname{res}\lambda(c) \tag{6.17}$$

where

$$\alpha_k = m_1 + \cdots + m_k$$

$$\beta_k = s_1 + \cdots + s_k.$$

Now let us observe that the operator $J(j, m_j, s_j)$, defined by (5.17), contains $A(p_k, m_j, s_j)$ as a factor if and only if $j \leqq k \leqq n - 1$. Thus the factors of the form $A(p_k, m, s)$ in the product (6.5) are precisely those of the form $A(p_k, m_j, s_j)$, $1 \leqq j \leqq k$. It therefore follows from (6.9) that

$$W(\lambda)T(c, \lambda)W(\lambda)^{-1} = \frac{\operatorname{res}\lambda(c)}{\lambda(c)} T(c, \lambda).$$

Hence $W(\lambda)T(c, \lambda)W(\lambda)^{-1} = T(c, \operatorname{res}\lambda)$.

We shall now prove (6.7) for $a = p_j$ where $1 \leqq j \leqq n-2$. Since $1 \leqq j \leqq n-2$, we see from (6.16) that $T(p_j, \operatorname{res}\lambda) = T(p_j, \epsilon)$. Thus it will suffice to show that

$$W(\lambda)T(p_j,\lambda) = T(p_j,\epsilon)W(\lambda). \tag{6.18}$$

Let $\lambda = (m_1, \cdots, m_n; s_1, \cdots, s_n)$. Then by (6.16),

$$T(p_j,\lambda) = T(p_j, m_j - m_{j+1}, s_j - s_{j+1})$$

and by (6.14)

$$J(k,\lambda)T(p_j,\lambda) = T(p_j,\lambda)J(k,\lambda),\quad j+1 < k \leqq n-1. \tag{6.19}$$

From (6.15) and (6.13) we see that

$$J(j+1,\lambda)T(p_j,\lambda) = T(p_j, m_j, s_j)J(j+1,\lambda). \tag{6.20}$$

Using (6.13), (6.15), and (6.14) we now find that

$$J(j,\lambda)T(p_j, m_j, s_j) = T(p_j,\epsilon)J(p,\lambda). \tag{6.21}$$

By similar computations with (6.13), (6.14), and (6.15) we also find that

$$J(k,\lambda)T(p_j,\epsilon) = T(p_j,\epsilon)J(k,\lambda),\quad 1 \leqq k < j. \tag{6.22}$$

Finally, we observe that (6.18) is implied by equations (6.19) through (6.22).

(2) If (6.8) holds for q_1, q_2 in S, then using the fact that $(q_1q_2)\lambda = q_1(q_2\lambda)$ for all λ we see that it is also valid for q_1q_2. It therefore suffices to prove (6.8) for the generators, p_k of S. Suppose then that k is fixed and let $p = p_k$. Then $R(a,\lambda) = R(a,p\lambda)$ if and only if

$$W(\lambda)T(a,\lambda)W(\lambda)^{-1} = W(p\lambda)T(a,p\lambda)W(p\lambda)^{-1}.$$

This condition is equivalent, however, to the condition that

$$W(p\lambda)^{-1}W(\lambda)T(a,\lambda) = T(a,p\lambda)W(p\lambda)^{-1}W(\lambda).$$

To prove this it will suffice, in view of (4.2), to show that $A(p,\lambda) = W(p\lambda)^{-1}W(\lambda)$. We shall prove the equivalent condition

$$W(p\lambda) = W(\lambda)A(p,\lambda)^{-1}. \tag{6.23}$$

Now $A(p_k,\lambda) = A(p_k, m_k - m_{k+1}, s_k - s_{k+1})$ and hence by Lemma 3

$$\begin{aligned} A(p_k,\lambda)^{-1} &= A(p_k, m_{k+1}, s_{k+1})A(p_k, -m_k, -s_k) \\ &= A(p_k, -m_k, -s_k)A(p_k, m_{k+1}, s_{k+1}). \end{aligned} \tag{6.24}$$

Thus $J(n-1,\lambda)A(p_{n-1},\lambda)^{-1}=J(n-1,p_{n-1}\lambda)$, and when $1\leqq j\leqq n-2$, $J(j,\lambda)=J(j,p_{n-1}\lambda)$. It follows from (6.5) that (6.23) holds when $k=n-1$. Now suppose $1\leqq k\leqq n-2$. Then

$$J(k+1,m_{k+1},s_{k+1})A(p_k,m_{k+1},s_{k+1})=J(k,m_{k+1},s_{k+1}).$$

It follows from (6.24) that

$$J(k,\lambda)J(k+1,\lambda)A(p_k,\lambda)^{-1}=J(k,m_k,s_k)J(k,m_{k+1},s_{k+1})A(p_k,-m_k,-s_k).$$

Furthermore, the corollary to Theorem 2 shows that $J(k,m_k,s_k)$ commutes with $J(k,m_{k+1},s_{k+1})$, and since $J(k,m_k,s_k)A(p_k,-m_k,-s_k)=J(k+1,m_k,s_k)$ we see that

$$J(k,\lambda)J(k+1,\lambda)A(p_k,\lambda)^{-1}=J(k,p_k\lambda)J(k+1,p_k\lambda). \tag{6.25}$$

In addition, $A(p_k,\lambda)^{-1}$ commutes with $J(j,\lambda)$ when $k+1<j\leqq n-1$, and $J(j,\lambda)=J(j,p_k\lambda)$ whenever $|k-j|>1$. From these observations and (6.25) we conclude that $W(p_k\lambda)=W(\lambda)A(p_k,\lambda)^{-1}$.

7. Lemmas on analytic continuation of certain operators. The results of this section as well as those of the next will be used in Section 9 to construct uniformly bounded representations of G. The main idea of this construction is to continue the operators $R(a,\lambda)$, of the normalized principal series, analytically in the parameters specifying λ.

We begin by recalling some standard definitions and results concerning analytic operator valued functions. Let H be a Hilbert space and $s\to P(s)$ a function defined in a domain D of the complsx plane whose values are bounded operators on H. Then $P(\cdot)$ is said to be analytic in D if $s\to(P(s)f,g)$ is analytic in D for all f, g in H. By a well known theorem of Hille ([7]; p. 53) such a function is continuous in the uniform operator norm and has a complex derivative, again in the uniform norm. From this it follows that if $P(\cdot)$ and $Q(\cdot)$ are analytic operator valued functions in D, then so is their product $s\to P(s)Q(s)$.

LEMMA 17. *Let $\gamma(m,s)$ be defined by (3.5) and $s=a+ib$. Then there is a constant K independent of m and s such that*

$$\frac{1}{|\gamma(m,s)|}<K(1+|m|+|b|)^{1-a},\quad -1<a<1. \tag{7.1}$$

Proof. Let $c=\dfrac{|m|+2}{2}$ and $z=\dfrac{s}{2}=x+iy$. Then

$$\frac{i^{|m|}\pi 2^s}{\gamma(m,s)}=\frac{\Gamma(c-z)}{\Gamma(c-1+z)}=(c-1+z)\frac{\Gamma(c-z)}{\Gamma(c+z)}.$$

To estimate the factor $\frac{\Gamma(c-z)}{\Gamma(c+z)}$ we use Stirling's formula (see Titchmarsh [10; p. 151]) which implies that $\log \Gamma(w) - (w-\frac{1}{2}) \log w + w$ is uniformly bounded in any sector of the form $-\pi + \epsilon \leqq \arg w \leqq \pi - \epsilon$ (for fixed $\epsilon > 0$) the logarithms being principal values. In this connection we also make use of the following easily verified estimates.

$$(7.2) \qquad |(c-z-\tfrac{1}{2})\log\left(\frac{c-z}{c-iy}\right)| \leqq |x|, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

$$(7.3) \qquad |(c+z-\tfrac{1}{2})\log\left(\frac{c+z}{c+iy}\right)| \leqq |x|, \quad -\tfrac{1}{2} < x < \tfrac{1}{2}.$$

From (7.2), (7.3), and Stirling's formula we find that

$$(7.4) \qquad \begin{aligned} &\log \Gamma(c-z) - \log \Gamma(c+z) \\ &= -2z \log |c-iy| + i(2c-1)\arg(c-iy) + 2z + B(z) \end{aligned}$$

where $B(z)$ is uniformly bounded in $-\frac{1}{2} < Re(z) < \frac{1}{2}$. At this point it is a straighforward matter to derive (7.1) from (7.4).

In the next three lemmas we shall be concerned with the analytic continuation of various combinations of the operators $A(m,s)$ and $B(m,s)$ treated in the last part of Section 5 (cf. equations (5.9), (5.10), and Lemma 11.)

LEMMA 18. *Let m be a fixed integer and $L_2 = L_2(v)$ where v is one complex variable. Then the operator $A(m,s)\cdot B(m,s)$ initially defined on L_2 for $Re(s) = 0$ can be continued into the strip $0 \leqq Re(s) < 1$ in such a way that*

(1) *for each f, g in L_2 the function*

$$s \to (A(m,s)B(m,s)f, g)$$

is analytic in $0 < Re(s) < 1$ and continuous in $0 \leqq Re(s) < 1$.

(2) $$\| A(m,s)B(m,s) \| \leqq K_d(1 + |m| + |b|)^{1-a}$$

where $s = a + ib$, $0 \leqq a \leqq d < 1$, and $\|\cdot\|$ denotes the usual operator norm.

Remark. Part of the interest of this result lies in the fact that the natural continuations of $A(m,s)$ and $B(m,s)$ are unbounded operators for $0 < Re(s) < 1$, but nevertheless their product $A(m,s)B(m,s)$ remains bounded.

Proof. Let D denote the linear space of all bounded, compactly sup-

ported functions in L_2 which also vanish in a neighborhood of the origin. If f and g belong to D and $Re(s)=0$ let

$$\Phi(f,g,s)=(A(m,s)B(m,s)f,g).$$

Then since $A(m,s)^*=A(-m,\bar{s})$ (see Lemma 3) it follows that

$$\Phi(f,g,s)=(B(m,s)f,A(-m,\bar{s})g).$$

Next we shall define $\Phi(f,g,s)$ when $0<Re(s)<1$. To do this we first put

$$B(m,s)f(v)=\left(\frac{v}{|v|}\right)^{-m}|v|^{-s}f(v),\ 0\leqq Re(s)<1. \tag{7.5}$$

Then $B(m,s)f\in L_2$, and if $f(v)=0$ when $|v|<\delta$ it follows that $|B(m,s)f|\leqq\delta^{-1}|f|$. Hence, by dominated convergence, $s\to B(m,s)f$ is continuous in $0\leqq Re(s)<1$ as a function with values in L_2. By Lemma 3, $s\to A(-m,\bar{s})g$ is also continuous as a function from $0\leqq Re(s)<1$ into L_2. Thus setting

$$\Phi(f,g,s)=(B(m,s)f,A(-m,\bar{s})g),\ 0\leqq Re(s)<1 \tag{7.6}$$

we see that $\Phi(f,g,s)$ is continuous in $0\leqq Re(s)<1$. Furthermore, $\Phi(f,g,s)$ is uniformly bounded in any strip of the form $0\leqq Re(s)\leqq d$, $0<d<1$. For the explicit form of the Fourier transform of $A(-m,\bar{s})g$, which is given in the proof of Lemma 3, shows that

$$\|A(-m,\bar{s})g\|_2\leqq\|A(-m,a)g\|_2,\quad s=a+ib, 0\leqq a<1.$$

Thus in $0\leqq Re(s)\leqq d$

$$|\Phi(f,g,s)|\leqq\delta^{-1}\|f\|_2\cdot\sup_{0\leqq a\leqq d}\|A(-m,a)g\|_2<\infty. \tag{7.7}$$

The bound given in (7.7) is of course dependent on f and g.

We now suppose $0<Re(s)<1$, and in order to simplify the notation we shall as before write $[v]$ for $\frac{v}{|v|}$. Then from (7.6), (7.5), (5.9), and the fact that $\bar{\gamma}(-m,\bar{s})=(-1)^m\gamma(m,s)$ we find that

$$\gamma(m,s)\Phi(f,g,s)=\int[v]^{-m}|v|^{-s}f(v)\,dv\int[z]^m|z|^{-2+s}\bar{g}(v+z)\,dz.$$

In the integral with respect to z, we make the transformation $z\to vz-v$ and obtain

$$\gamma(m,s)\Phi(f,g,s)=\int f(v)\,dv\int[z-1]^m|z-1|^{-2+s}\bar{g}(zv)\,dz. \tag{7.8}$$

Thus

$$|\gamma(m,s)\Phi(f,g,s)|\leqq\int|f(v)|\,dv\int|z-1|^{-2+s}|g(zv)|\,dz$$

where $s = a + ib$ and $0 < a < 1$. Now setting

$$h(v) = \int | z - 1 |^{-2+a} | g(zv) | \, dz$$

and using Minkowski's inequality for integrals we find that

$$\| h \|_2 \leqq \int | z - 1 |^{-2+a} [\int | g(zv) |^2 \, dv]^{\frac{1}{2}} \, dz = K_a' \| g \|_2$$

where $K_a' = \int | z - 1 |^{-2+a} | z |^{-1} \, dz$ and is uniformly bounded in any closed interval contained in $(0, 1)$. In fact K_a' is continuous as a function of a in the open interval $(0, 1)$. Thus

$$\int | f(v) | \, dv \int | z - 1 |^{-2+a} | g(zv) | \, dz \leqq K_a' \| f \|_2 \| g \|_2$$

and setting

$$F(s) = \int f(v) dv \int [z - 1]^m | z - 1 |^{-2+s} \bar{g}(zv) dz, \; 0 < Re(s) < 1$$

we find that the integrals defining F are absolutely convergent in $0 < Re(s) < 1$. Moreover, because of the boundedness of K_a', Fubini's theorem shows that

$$\int_\Delta F(s) \, dx = 0$$

for every triangle Δ contained in the strip $0 < Re(s) < 1$. Thus F is analytic in $0 < Re(s) < 1$. Now in view of the analyticity of $1/\gamma(m, s)$ and (7.8) it follows that $\Phi(f, g, s)$ is analytic in $0 < Re(s) < 1$. If $Re(s) = 0$ and s is fixed, the definition of Φ shows that

$$f, g \rightarrow \Phi(f, g, s)$$

is a sesqui-linear form on D. Thus, since $s \rightarrow \Phi(f, g, s)$ is analytic in $0 < Re(s) < 1$ and continuous in $0 \leqq Re(s) < 1$, for fixed f, g in D, we see that

$$f, g \rightarrow \Phi(f, g, s)$$

is also sesqui-linear when $0 < Re(s) < 1$. This follows because a function G is identically 0 if it is analytic in $0 < Re(s) < 1$, continuous in $0 \leqq Re(s) < 1$, and vanishes when $Re(s) = 0$.

Next we shall obtain a bound for Φ. First of all we see from Lemma 17 and the estimates given above that

$$(7.9) \qquad | \Phi(f, g, s) | \leqq K_a (1 + | m | + | b |)^{1-a} \| f \|_2 \| g \|_2$$

where $s = a + ib$, $0 < a < 1$, and K_a is independent of f and g; however, K_a

does not remain bounded as $a \to 0$. In order to get around this difficulty we shall use a standard Phragmén-Lindelöf argument.

We fix d so that $0 < d < 1$. Then in (7.9) we may assume without loss of generality that $K_d \geqq 1$. Now consider the function Φ_1 defined by

$$\Phi_1(s) = K_d^{-1}(1 + |m| + s)^{a-1}(\| f \|_2 \| g \|_2)\Phi(f, g, s), \quad 0 \leqq Re(s) < 1.$$

When $Re(s) = 0$, the operators $B(m, s)$ and $A(-m, \bar{s})$ are unitary. Hence from (7.6) we find that $|\Phi(f, g, s)| \leqq \| f \|_2 \| g \|_2$, $Re(s) = 0$. Therefore

$$|\Phi_1(s)| \leqq 1, \quad Re(s) = 0.$$

And from (7.9) it follows that

$$|\Phi_1(s)| \leqq 1, \quad Re(s) = d.$$

In addition, (7.7) shows that $\Phi(f, g, s)$ is uniformly bounded in $0 \leqq Re(s) \leqq d$; thus Φ_1 is also uniformly bounded in $0 \leqq Re(s) \leqq d$. We conclude that $|\Phi_1(s)| \leqq 1$, $0 \leqq Re(s) \leqq d$. This shows that (7.9) can be replaced by the stronger estimate

$$|\Phi(f, g, s)| \leqq K_d(1 + |m| + |b|)^{1-a} \| f_2 \| \| g \|_2, \qquad s = a + ib, \; 0 \leqq a \leqq d < 1. \tag{7.10}$$

Since (7.10) holds for a dense class of functions f, g there is for each s in $0 < Re(s) < 1$, a unique sesqui-linear form

$$f, g \to \Phi_1(f, g, s)$$

defined for all f, g in L_2 which extends

$$f, g \to \Phi(f, g, s)$$

and satisfies (7.10). Because the uniform limit of continuous (analytic) functions is still continuous (analytic) the function

$$s \to \Phi_1(f, g, s)$$

is continuous in $0 \leqq Re(s) < 1$ and analytic in $0 < Re(s) < 1$, for each f, g in L_2. Let $A(m, s)B(m, s)$ denote the operator on L_2 which is defined by $\Phi_1(\cdot, \cdot, s)$. It is then apparent that the operator valued function $s \to A(m, s)B(m, s)$ satisfies (1) and (2). This concludes the proof of the lemma.

We now define another operator $C(m, m', s, s')$ on $L_2 = L_2(v)$, where v is one complex variable, by setting

$$\begin{aligned}(7.11)\qquad & C(m,m',s,s') \\ & = A(m,s)B(m+m',s+s') - B(m+m',s+s')A(m,s).\end{aligned}$$

Here we assume that $Re(s) = Re(s') = 0$ and that m and m' are integers.

LEMMA 19. *Let m, m' and s' be fixed. Then the operator $C(m,m',s,s')$, initially defined on L_2 for $Re(s)=0$, can be continued into the strip $-1 < Re(s) < 1$ in such a way that*

(1) *for each f, g in L_2 the function*

$$s \rightarrow (C(m,m',s,s')f,g)$$

is analytic in $-1 < Re(s) < 1$, and

(2) $\| C(m,m',s,s') \| \leqq K_a(1 + |m| + |m'| + |b| + |b'|)^{2-a}$

where $\|\cdot\|$ denotes the usual operator norm, $s = a + ib$, $s' = ib'$, and $-1 < a < 1$.

Proof. The proof is similar to part of the proof of Lemma 18, and we shall therefore omit some of the details. We begin with functions f, g both of which are bounded, vanish outside a bounded set, and are 0 in a neighborhood of the origin. For such functions f, g we put

$$\begin{aligned}(7.12)\qquad (f,g,s) &= (B(m+m',s+s')f, A(-m,\bar{s})g) \\ &\qquad - (A(m,s)f, B(-m,-m',-\bar{s}+\bar{s}')g)\end{aligned}$$

where $0 \leqq Re(s) < 1$. Then

$$\Phi(f,g,s) = (C(m,m',s,s')f,g)$$

when $Re(s) = 0$. On the other hand when $0 < Re(s) < 1$ we find, by rewriting the integrals, that

$$\begin{aligned}&\gamma(m,s)\Phi(f,g,s) \\ &= \int\int [z-v]^m\, |z-v|^{-2+s} f(v)\bar{g}(z)\,([v]^{-m-m'}\,|v|^{-s-s'} - [z]^{-m-m'}\,|z|^{-s-s'})\,dvdz.\end{aligned}$$

Now by means of the transformation $v \rightarrow zv$, $z \rightarrow z$ we obtain the relation

$$\begin{aligned}(7.13)\qquad &\gamma(m,s)\Phi(f,g,s) \\ &= [z]^{-m'}\,|z|^{-s'} \int\int [1-v]^m\, |1-v|^{-2+s} f(zv)\bar{g}(z)\,([v]^{-m-m'}\,|v|^{-s-s'} - 1)\,dvdz\end{aligned}$$

which holds when $0 < Re(s) < 1$. We shall show next that the integral in (7.13) converges absolutely whenever f, g belong to L_2 and $-1 < Re(s) < 1$.

Let I_1 denote the integral over the set where $|v-1| \leqq \frac{1}{2}$ and I_2 the integral over the complementary set. To estimate I_1 we use the inequality

$$|[v]^{-m-m'}|v|^{-s-s'}-1| \leqq K(1+|m+m'|+|b+b'|)|v-1| \tag{7.14}$$

which is valid for $|v-1| \leqq \frac{1}{2}$, $s=a+ib$, $s'=ib'$, and $-1<a<1$. It follows from (7.14) that

$$|I_1| \leqq K(1+|m+m'|+|b+b'|)\int\int_{|v-1|\leqq\frac{1}{2}} |f(zv)\bar{g}(z)|\,|v-1|^{-1+a}dvdz.$$

By Minkowski's inequality for integrals,

$$|I_1| \leqq K(1+|m+m'|+|b+b'|)\|f\|_2\|g\|_2\int\int_{|v-1|\leqq\frac{1}{2}}|v-1|^{-1+a}|v|^{-1}dv$$
$$=K_a(1+m+m'|+|b+b'|)\|f\|_2\|g\|_2, -1<a.$$

Now consider I_2. Notice that

$$|1-[v]^{-m-m'}|v|^{-s-s'}| \leqq 1+|v|^{-a},\ s=a+ib$$

and that

$$|v-1|^{-2+a} \leqq K(1+|v|)^{-2+a},\ |v-1| \geqq \tfrac{1}{2}.$$

Hence

$$|I_2| \leqq K(1+|m+m'|+|b+b'|)$$
$$\int\int_{|v-1|\geqq\frac{1}{2}} |f(vz)g(z)|(1+|v|)^{-2+a}(1+|v|^{-a})dvdz.$$

Moreover

$$\int\int|f(vz)g(z)|(1+|v|)^{-2+a}(1+|v|^{-a})dvdz$$
$$\leqq \|f\|_2\|g\|_2\int|v|^{-1}(1+|v|)^{-2+a}(1+|v|^{-a})dv$$

and $\int|v|^{-1}(1+|v|)^{-2+a}(1+|v|^{-a})dv<\infty$ if $a<1$. These estimates show that the integral in (7.13) is bounded by

$$K_a(1+|m+m'|+|b+b'|)\|f\|_2\|g\|_2$$

for $-1<a<1$ and all f,g in L_2. Furthermore, the above argument, in which we obtained the estimates for $|I_1|$ and $|I_2|$, can easily be modified to show that the integral in (7.13) is analytic as a function of s in $-1<Re(s)<1$. From (7.12) it follows that $\Phi(f,g,s)$ is continuous in $0 \leqq Re(s)<1$. Hence we may use the integral in (7.13) to define

$\gamma(m,s)\Phi(f,g,s)$ for $-1 < Re(s) < 1$. Then from the above estimates and Lemma 17 we conclude that

$$(7.15) \qquad |\Phi(f,g,s)| \leqq K_a(1+|m|+|m'|+|b|+|b'|)^{2-a} \|f\|_2 \|g\|_2.$$

for $-1 < a < 1$ and all f, g in L_2. Thus statements (1) and (2) of the lemma are satisfied by the operator $C(m,m',s,s')$ which is defined by the form $\Phi(\cdot,\cdot,s)$.

Next we define another operator by setting

$$(7.16) \qquad E(m,m',s,s') = A(m,s)B(m+m',s+s')A(-m,-s)B(-m,-s).$$

Here we assume that m and m' are integers and that $Re(s) = Re(s') = 0$.

LEMMA 20. *Let m, m' and s' be fixed. Then the operator $E(m,m',s,s')$, initially defined on L_2 for $Re(s) = 0$, can be continued into the strip $-1 < Re(s) < 1$ in such a way that*

(1) *for each f, g in L_2, the function*

$$s \to (E(m,m',s,s')f,g)$$

is analytic in $-1 < Re(s) < 1$, and

(2) $\|E(m,m',s,s')\| \leqq K_a(1+|m|+|m'|+|b|+|b'|)^3$

where $s = a+ib$, $s' = ib'$, and $-1 < a < 1$.

Proof. If $Re(s) = 0$, (7.11) shows that

$$A(m,s)B(m+m',s+s') = C(m,m',s,s') + B(m+m',s+s')A(m,s).$$

Hence rewriting (7.16) we find that

$$E(m,m',s,s') = C(m,m',s,s')A(-m,-s)B(-m,-s) + B(m',s').$$

In view of the two preceding lemmas we may now extend the definition of $E(m,m',s,s')$ by setting

$$(7.17) \qquad E(m,m',s,s') = C(m,m',s,s')A(-m,-s)B(-m,-s) + B(m',s'),$$
$$-1 < Re(s) \leqq 0.$$

On the other hand when $Re(s) = 0$, it is possible to rewrite (7.16) in another way. For it is easy to see that

$$A(m,s)B(m+m',s+s') \cdot C(-m,-m',-s,-s')$$
$$= E(m,m',s,s')B(-m',-s') - I.$$

Hence

$$E(m,m',s,s')$$
$$=A(m,s)B(m,s)\cdot B(m',s')C(m,m',s,s')B(m',s')+B(m',s').$$

By Lemmas 18 and 19, the right side of this equation is well defined for $0\leqq Re(s)<1$. We shall put

$$(7.18)\quad E(m,m',s,s')$$
$$=A(m,s)B(m,s)\cdot B(m',s')C(-m,-m',-s,-s')B(m',s')+B(m',s'),$$
$$0\leqq Re(s)<1.$$

To prove (1) let us suppose f and g are fixed functions in L_2. Let

$$\Phi_+(s)=(E(m,m',s,s')f,g),\quad 0\leqq Re(s)<1.$$

Also let

$$P(s)=A(m,s)B(m,s)\quad\text{and}\quad Q(s)=B(m',s')C(-m,-m',-s,-s')B(m',s').$$

It follows from Lemmas 18 and 19 that $P(s)$ and $Q(s)$ are analytic in $0<Re(s)<1$, and hence so is their product $s\to P(s)Q(s)$. From (7.18) we conclude that Φ_+ is also analytic in $0<Re(s)<1$. Now suppose $Re(s_0)=0$. Then

$$(P(s)Q(s)f,g)=(P(s)Q(s_0)f,g)+(Q(s)f-Q(s_0)f,P(s)^*g)$$

and Lemma 18 shows that $(P(s)Q(s_0)f,g)\to(P(s_0)Q(s_0)f,g)$ as $s\to s_0$. On the other hand, it also shows that $\| P(s)\|$ is uniformly bounded in every compact subset of $0\leqq Re(s)<1$. Hence, since $Q(s)$ is uniformly continuous

$$|(Q(s)f-Q(s_0)f,P(s)^*g)|\leqq K\,\|Q(s)-Q(s_0)\|\,\|f\|_2\,\|g\|_2\to 0$$

as $s\to s_0$. It follows that Φ_+ is continuous in $0\leqq Re(s)<1$. Next we put

$$\Phi_-(s)=(E(m,m',s,s')f,g),\quad -1<Re(s)\leqq 0.$$

Then, by arguments similar to those just given, it follows that Φ_- is analytic in $-1<Re(s)<0$ and continuous in $-1<Re(s)\leqq 0$. Since $\Phi_+(s)=\Phi_-(s)$ when $Re(s)=0$, Φ_+ and Φ_- are analytic continuations of each other. This proves (1). Finally, we observe that (2) follows from (7.17), (7.18), the fact that $B(m',s')$ is unitary, and the bounds given in Lemmas 18 and 19.

8. Some lemmas on several complex variables. The first result we prove here is similar to well known theorems (cf. Bochner and Martin, *Several*

Complex Variables) on the analytic continuation of functions of several complex variables, but deals with functions initially defined on a set which is considerably thinner than those which appear to have been treated in the literature.

LEMMA 21. *Let $G(z_1, z_2, \cdots, z_m)$ be a given function of m variables $(z_j = x_j + iy_j)$, initially defined and continuous on the set where $Re(z_j) = 0$, $j = 1, 2, \cdots, m$. Suppose that for every fixed j, and fixed $iy_1, iy_2, \cdots, iy_{j-1}, iy_{j+1}, \cdots, iy_m$*

$$G(iy_1, \cdots, iy_{j-1}, z_j, iy_{j+1}, \cdots, iy_m)$$

as a function of z_j is analytically extendable into the strip $|Re(z_j)| < 1$ and satisfies $|G| \leqq 1$ there. Then G is extendable into the tube

$$|x_1| + |x_2| + \cdots + |x_m| < 1 \quad (y\text{'s arbitrary})$$

is analytic there, as a function of the m variables, and satisfies $|G| \leqq 1$ there.

Proof. Let $G(iy_1, iy_2, \cdots, iy_m) = G(iy)$ and $D_j = \frac{1}{i}\frac{\partial}{\partial y_j}$. It then follows easily from our assumptions that for each j

$$\sup_{y} |D_j^n G(iy)| \leqq n!, \quad n = 0, 1, 2, \cdots. \tag{8.1}$$

Let $H(z) = (2 - z_1)^{-2}(2 - z_2)^{-2} \cdots (2 - z_m)^{-2}$. Then

$$D_j^k H(iy) = \frac{(k+1)!}{(2 - iy_j)^k} \cdot H(iy).$$

Now putting $F(iy) = G(iy)H(iy)$, we find that

$$D_j^n F(iy) = H(iy) \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{(k+1)!}{(2 - iy_j)^k} \cdot D_j^{n-k} G(iy).$$

Hence by (8.1) and simple estimates

$$|D_j^n F(iy)| \leqq (n+1)! \, |H(iy)|. \tag{8.2}$$

Let $f(\lambda) = \int e^{i\langle y, \lambda\rangle} F(iy)\, dy$ where $\langle y, \lambda\rangle = y_1\lambda_1 + y_2\lambda_2 + \cdots + y_m\lambda_m$. Then integrating by parts, we find that

$$\lambda_j^n f(\lambda) = \int e^{i\langle y, \lambda\rangle} D_j^n F(iy)\, dy.$$

Thus in view of (8.2) and the fact that H is integrable, there is a constant A such that

$$\sup_{\lambda} |\lambda_j^n f(\lambda)| \leqq A(n+1)!, \quad n = 0, 1, 2, \cdots. \tag{8.3}$$

It follows that F can be expressed in terms of f by means of the Fourier inversion formula. Thus

$$F(iy) = \frac{1}{(2\pi)^m} \int e^{-i\langle y,\lambda\rangle} f(\lambda)\, d\lambda.$$

Next we set

$$F(x+iy) = \frac{1}{(2\pi)^m} \int e^{-\langle x+iy,\lambda\rangle} f(\lambda)\, d\lambda.$$

We shall show that this integral converges absolutely whenever $x+iy$ is in the tube described above. Let x be fixed and such that $\sum_j |x_j| = \mu < 1$. Then for each λ, $|\langle x, \lambda\rangle| \leqq \mu \max |\lambda_j|$. Thus

$$\int e^{-\langle x,\lambda\rangle} |f(\lambda)|\, d\lambda \leqq \sum_n \frac{\mu^n}{n!} \int \max |\lambda_j|^n |f(\lambda)|\, d\lambda.$$

Now using (8.3), we see that

$$\int \max |\lambda_j|^n |f(\lambda)|\, d\lambda \leqq A(n+m+2)! \int \frac{d\lambda}{1+\max |\lambda_j|^{m+1}}$$

$$\leqq K(n+m+2)! < \infty.$$

Since the series

$$\sum_n K \cdot \frac{(n+m+2)!}{n!} \cdot \mu^n$$

is convergent, it follows that

$$\int e^{-\langle x,\lambda\rangle} |f(\lambda)|\, d\lambda < \infty.$$

Thus the integral in (8.4) is absolutely convergent, and the function $F(z) = F(z_1, z_2, \cdots, z_m)$, which it defines, is analytic in the tube $\sum |Re(z_j)| < 1$. We now set

$$G(z) = \frac{F(z)}{H(z)}, \quad \sum |Re(z_j)| < 1.$$

Then G is an analytic extension of our original function into the tube. To show that $|G| \leqq 1$ we suppose that there is some point z_0 such that $|G(z_0)| > 1$. We then consider the function

$$\tilde{G}(iy) = \frac{A}{w - G(iy)}$$

where $w = G(z_0)$ and A is chosen so that $\operatorname{Sup}_y |\tilde{G}(iy)| \leqq 1$. The function $\tilde{G}$ then satisfies our original conditions on G but obviously cannot be extended so as to be analytic at z_0. Hence $|G| \leqq 1$.

LEMMA 22. *Let T be a linear map of $\boldsymbol{C}^m$ to $\boldsymbol{C}^p$ such that $T(\boldsymbol{R}^m) = \boldsymbol{R}^p$, and let Γ be the tube in $\boldsymbol{C}^m$ consisting of all $z = x + iy$ such that $\sum |x_j| < 1$. Suppose G is an analytic function defined on Γ with the property that $G(iy) = G(iy')$ whenever $Ty = Ty'$. Then there is a unique analytic function F defined on $T(\Gamma)$ such that*

$$F(Tz) = G(z), \quad z \in \Gamma. \tag{8.5}$$

Proof. We may suppose the null space of T has dimension k where $k > 0$. Then since $T(\boldsymbol{R}^m) \subset T(\boldsymbol{R}^p)$ there is a linear map $w \to \tilde{w}$ of $\boldsymbol{R}^k$ onto the null space of T which sends real vectors into real vectors. Fix $v \in \boldsymbol{R}^k$ and for $a \in \Gamma$, let $H(z) = G(z + i\tilde{v}) - G(z)$. Then H is analytic in Γ and $H(iy) = 0$ for all $y \in \boldsymbol{R}^m$. Since Γ is a tube and is both open and connected, it follows by analytic continuation that

$$G(z + i\tilde{v}) = G(z), \quad z \in \Gamma. \tag{8.6}$$

Now fix $z \in \Gamma$ and let

$$\Gamma_z = \{w \in \boldsymbol{R}^k : \tilde{w} + z \in \Gamma\}.$$

Then Γ_z is an open connected tube in $\boldsymbol{R}^k$ which contains the set $Re(w) = 0$. In Γ_z we define a function K by setting $K(w) = G(z + i\tilde{w}) - G(z)$. Then K is analytic in Γ_z and by (8.6) $K(iv) = 0$ for all $v \in \boldsymbol{R}^k$. Thus, by analytic continuation, we see that $G(z + \tilde{w}) = G(z)$ for all $w \in \Gamma_z$. In other words, $G(z) = G(z')$ whenever z and z' belong to Γ and $Tz = Tz'$. It follows that there is a unique well defined function F on $T(\Gamma)$ which satisfies (8.5). Furthermore, our assumptions on T imply that $T(\Gamma)$ is an open convex tube in $\boldsymbol{C}$. Now let s' be a fixed vector in $T(\Gamma)$. Then there exists a vector z' in Γ and a non-singular linear map L of $\boldsymbol{C}^p$ into $\boldsymbol{C}^m$ such that $Ls' = z'$ and TL is the identity on $\boldsymbol{C}^p$. If s is sufficiently close to s', $F(s) = G(Ls)$. It follows that F is analytic in a neighborhood of s' and hence that F is analytic in $T(\Gamma)$.

In our application we shall not use Lemma 21 directly but a slight variant of it. We shall be concerned with a function $F(s_1, s_2, \cdots, s_n)$ initially defined in a subset of the complex hyperplane $s_1 + s_2 + \cdots + s_n = 0$. Our problem as before, will be to extend F so that it is analytic in a larger set. We shall say a function defined on a subset of $s_1 + s_2 + \cdots + s_n = 0$ is analytic if it is analytic in the usual sense as a function of the $n - 1$ complex variables $s_1, s_2, \cdots, s_{n-1}$ (with $s_n = -s_1 - s_2 - \cdots - s_{n-1}$). Before stating our result it will be convenient to introduce some notation. Let $0 < d \leqq 1$, $s_j = a_j + ib_j$ and $B(d)$ denote the smallest convex set in the real hyper-

plane $a_1 + a_2 + \cdots + a_n = 0$ which contains the points $(a, -a, 0, \cdots, 0)$, $-d < a < d$, and all permutations of these points.

Now let $\mathfrak{J}(d)$ denote the tube in $s_1 + s_2 + \cdots + s_n = 0$ whose basis is $B(d)$, i.e., the set of all points $(a_1 + ib_1, \cdots, a_n + ib_n)$ where $(a_1, a_2, \cdots, a_n) \in B(d)$ and $b_1 + b_2 + \cdots + b_n = 0$, but otherwise the b's are arbitrary. Then $\mathfrak{J}(d)$ is an open convex subset of $s_1 + s_2 + \cdots + s_n = 0$, and with these conventions our result can be formulated in the following way.

LEMMA 23. *Let F be defined and continuous on the set where $s_1 + s_2 + \cdots + s_n = 0$ and $a_j = 0$, $j = 1, 2, \cdots, n$. Moreover, suppose that for every pair j, k, $1 \leqq j < k \leqq n$ and fixed $ib_1, ib_2, \cdots, ib_n$ (such that $\sum b_j = 0$) the function*

$$F_{jk}(s) = F(ib_1, \cdots, ib_j + s, \cdots, ib_k - s, \cdots, ib_n)$$

as a function of the one complex variable $s = a + ib$ has an analytic extension into the strip $-d < Re(s) < d$ and $|F_{jk}(s)| \leqq 1$ there. Then F has an analytic extension to the tube $\mathfrak{J}(d)$ which satisfies the condition $|F| \leqq 1$.

Proof. Set $m = \dfrac{n(n-1)}{2}$, label the coordinates of points $z = x + iy$ in $\boldsymbol{C}^m$ with two indices jk, $1 \leqq j < k \leqq n$, and let $\tilde{T}$ be the linear map of $\boldsymbol{C}^m$ onto the hyperplane $s_1 + s_2 + \cdots + s_n = 0$ which is given by the equations

$$s_j = d \sum_{k>j} z_{jk} - d \sum_{k<j} z_{kj}, \quad 1 \leqq j \leqq n.$$

Put $G(iy) = F(i\tilde{T}y)$. Then it is easily seen that Lemma 22 applies to G and hence that G can be extended analytically into the tube Γ whose basis is the set $\sum |x_j| < 1$. Now let $p = n - 1$ and T be the map $\tilde{T}$ followed by the projection $(s_1, s_2, \cdots, s_n) \to (s_1, s_2, \cdots, s_{n-1})$. Applying Lemma 22 and the observation that $\tilde{T}(\Gamma) = \mathfrak{J}(d)$, we easily obtain the conclusion of Lemma 24.

9. Analytic continuation of the normalized principal series. In this section we consider the problem of the analytic continuation of the representations $a \to R(a, \lambda)$ which are intially defined for unitary characters by (6.6). We shall obtain, in the end, a larger family of uniformly bounded representations, some of which are again unitary (the complementary series) and others which are not.

Let $\mathfrak{J}$ be the 'tube' $\mathfrak{J}(1)$ which is described in the paragraph preceding

Lemma 23. We denote by Λ^* the set of all characters λ of the diagonal sub-group C of G having the form

$$(9.1) \qquad \lambda(c) = \prod_{j=1}^{n} \left(\frac{c_j}{|c_j|}\right) |c_j|^{s_j}, \quad c \in C$$

where $m_1, m_2, \cdots, m_n$ are integers and $s_1, s_2, \cdots, s_n$ are complex numbers satisfying

(i) $0 \leqq \sum m_j < n$,

(ii) $(s_1, s_2, \cdots, s_n) \in \mathfrak{I}$.

We shall use the notation

$$\lambda = (m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$$

for the character given by (9.1).

It is clear that Λ^* contains the set of all continuous unitary characters of C, i. e., those which in addition to the above satisfy the condition $Re(s_j) = 0$, $j = 1, 2, \cdots, n$. As in Section 6, we shall denote the set of unitary characters by Λ.

A given complex or operator valued function

$$F(\lambda) = F(m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$$

defined on Λ^* will be called *analytic* if for fixed $m_1, m_2, \cdots, m_n$ it is complex analytic as a function of $s_1, s_2, \cdots, s_{n-1}$.

Having made these conventions we are now in a position to state our main theorem.

THEOREM 4. *Let G be the complex unimodular group of degree n, C its diagonal sub-group, Λ the collection of continuous unitary characters of C, and Λ^* the larger collection defined by* (9.1). *If $a \in G$ and $\lambda \in \Lambda$, let $R(a, \lambda)$ be the operator of the normalized principal series given by* (6.6). *Then for each fixed $a \in G$, the operator valued function*

$$\lambda \to R(a, \lambda)$$

initially defined on Λ can be extended into Λ^* in such a way that

(1) *it is analytic in Λ^*;*

(2) *for each fixed $\lambda \in \Lambda^*$*

$$a \to R(a, \lambda), \quad a \in G$$

is a continuous uniformly bounded representation of G on $L_2(V)$, V being the group of lower triangular unipotent matrices in G;

(3) *if S is the Weyl group, the representations $a \to R(a,\lambda)$ are invariant under S in the sense that*

$$R(a, q\lambda) = R(a, \lambda)$$

for all $a \in G$, $\lambda \in \Lambda^*$, and $q \in S$, where $q\lambda$ is the element of Λ^* given by

$$(q\lambda)(c) = \lambda(q^{-1}cq), \quad c \in C.$$

(4) *if the contragredient of a character in Λ^* or an operator on $L_2(V)$ is denoted by a prime then*

$$R(a, \lambda') = R(a, \lambda)'$$

for all $a \in G$ and all $\lambda \in \Lambda^*$.

Remarks. (i) In the proof of (2) we obtain an explicit but by no means best possible bound for the representation $a \to R(a,\lambda)$. Our estimate may be described as follows: Let $\lambda = (m_1, m_2, \cdots, m_m; s_1, s_2, \cdots, s_n) \in \Lambda^*$. Then there is a real number d such that $0 < d < 1$ and $-d < Re(s_j) < d$, $j = 1, 2, \cdots, n$. We show that

$$\operatorname*{Sup}_{a \in G} \| R(a, \lambda) \| \leqq K_d' \, | H(\lambda) |$$

where $\| \cdot \|$ denotes the usual operator norm, K_d' is independent of λ, and $H(\lambda)$ is the analytic function on Λ^* defined by (9.9).

(ii) Let $\lambda \in \Lambda^*$ and $a \in G$. Then $\lambda'(c) = \bar{\lambda}(c^{-1})$, $c \in C$ and $R(a,\lambda)'$ is the adjoint of $R(a,\lambda)^{-1}$. Thus λ is unitary if and only if $\lambda = \lambda'$, and in this case, statement (4) simply says that the representations of the normalized principal series are unitary. On the other hand, it is well known that G has another series of unitary representations, the so-called complementary series whose members are determined by certain non-unitary characters of C.

In the present situation, we obtain the complementary series, more precisely representations unitarily equivalent to the complementary series, in a very simple fashion. In fact the following result is an immediate consequence of (3) and (4) above.

COROLLARY. *Let $\lambda \in \Lambda^*$ and suppose there is an element $q \in S$ such that $\lambda' = q\lambda$. Then $a \rightarrow R(a,\lambda)$ is a unitary representation of G.*[8]

In the proof of the theorem we shall need a number of lemmas.

LEMMA 24. *Let $p = p_{n-1}$ where p_{n-1} is the element of the Weyl group defined by* (3.1). *For $Re(s) = 0$ and*

$$\lambda = (m_1, m_2, \cdots, m_n; ib_1, ib_2, \cdots, ib_n) \in \Lambda,$$

let

$$Q(\lambda, s) = R(p, m_1, \cdots, m_n; ib_1, \cdots, ib_{n-1} + s, ib_n - s). \tag{9.2}$$

Then for fixed λ, the operator $Q(\lambda, s)$ can be continued into the strip $-1 < Re(s) < 1$, in such a way that

(1) *for each f, g in $L_2(V)$, the function*

$$s \rightarrow (Q(\lambda, s)f, g)$$

is analytic in $-1 < Re(s) < 1$, and

(2) $\| Q(\lambda, s) \| \leqq K_a(1 + (\Sigma \mid m_j \mid + \mid b_j \mid) + \mid b \mid)^3$

where $s = a + ib$, $-1 < a < 1$.

Proof. By (6.6),

$$R(p,\lambda) = W(\lambda)T(p,\lambda)W(\lambda)^{-1} \text{ where } W(\lambda) = J(1,\lambda)J(2,\lambda) \cdots J(n-1,\lambda)$$

and

$$J(k,\lambda) = A(p_{n-1}, m_k, ib_k) \cdots A(p_{k+1}, m_k, ib_k) A(p_k, m_k, ib_k). \tag{9.3}$$

Let us put

$$W_1(\lambda) = J(1,\lambda) \cdots J(n-2,\lambda),\ T_1(p,\lambda) = J(n-1,\lambda)T(p,\lambda)J(n-1,\lambda)^{-1},$$

and

$$Q_1(\lambda, s) = T_1(p, m_1, \cdots, m_n; ib_1, \cdots, ib_{n-1} + s, ib_n - s). \tag{9.4}$$

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[8] That these representations are unitarily equivalent to the representations of the complementary series may be seen as follows: The two pairs of representations have the same characters as may be seen from formula (10.4) and the trace formula on page 115 of [3]. Thus it follows by a theorem of Harish-Chandra in [6] that the representations are untarily equivalent.

Then $R(p,\lambda)=W_1(\lambda)T_1(p,\lambda)W_1(\lambda)^{-1}$, and it follows from (9.3) that $W_1(\lambda)$ is independent of ib_{n-1}; in fact, $W_1(\lambda)=W_1(m_1,\cdots,m_{n-2};ib_1,\cdots,ib_{n-2})$. Thus

$$(9.5)\qquad Q(\lambda,s)=W_1(\lambda)Q_1(\lambda,s)W_1(\lambda),\ Re(s)=0.$$

Since $W_1(\lambda)$ is independent of s and unitary, it will suffice to prove statements (1) and (2) of the lemma for Q_1 instead of Q. In doing this, we first note that

$$\begin{aligned}T(p,\lambda)&=T(p,m_{n-1}-m_n,ib_{n-1}-ib_n)\\&=M(p,-m_n,-b_n)T(p,m_{n-1},ib_{n-1})\end{aligned}$$

where $M(p,-m_n,-ib_n)$ and $T(p,m_{n-1},ib_{n-1})$ are given by (6.11) and (6.12) respectively. To simplify the notation we set

$$m_{n-1}=m,\ b_{n-1}=b,\ m'=-m_{n-1}-m_n,\text{ and } b'=-b_{n-1}-b_n.$$

Then by the multiplicative properties of M

$$T(p,\lambda)=M(p,m+m',ib+ib')T(p,m,ib).$$

Thus $T_1(p,\lambda)=A(p,m,ib)B(p,m+m',ib+ib')T(p,m,ib)A(p,-m,-ib)$. Mcreover, (6.13) and (6.12) show that

$$\begin{aligned}T(p,m,ib)A(p,-m,-ib)&=A(p,-m,-ib)T(p,-m,-ib)\\&=A(p,-m,-ib)M(p,-m,-ib)T(p,\epsilon).\end{aligned}$$

Now setting

$$\begin{aligned}(9.6)\quad &L(m,m',ib,ib')\\&=L(m,m',ib,ib')\\&=A(p,m,ib)M(p,m+m',ib+ib')A(p,-m,-ib)M(p,-m,-ib)\end{aligned}$$

it follows that $T_1(p,\lambda)=L(m,m',ib,ib')T(p,\epsilon)$. Since $b'=-b_{n-1}-b_n$, i.e., since $b'=b_1+\cdots+b_{n-2}$, we see from (9.4) that

$$(9.7)\qquad Q_1(\lambda,s)=L(m,m',ib_{n-1}+s,ib')T(p,\epsilon),\ Re(s)=0.$$

Let us consider the operator $L(m,m',ib_{n-1},ib')$. To study its action we use (as in the proof of Lemma 3, Section3) the decomposition

$$L_2(V)=L_2(Z\times Z'),\quad Z=Z(n-1,n).$$

We recall that this representation of $L_2(V)$ arises from the expression of V as the semi-direct product ZZ' and that Z is isomorphic to the additive group of complex numbers. If $f \in L_2(V)$ we can therefore write $f(v) = f(z, z')$. Let us suppose that $f(z, z') = g(z)h(z')$ where $g \in L_2(Z)$ and $h \in L_2(Z')$. It is then easily verified that

$$(9.8) \qquad L(m, m', ib_{n-1}, ib')f(z, z') = E(m, m', ib_{n-1}, ib')g(z) \cdot h(z')$$

where $E(m, m', ib_{n-1}, ib')$ is the operator defined by (7.16).

Hence upon applying Lemma 20, we see that $E(m, m', ib_{n-1} + s, ib')$ can be continued so as to be analytic in $-1 < Re(s) < 1$. In addition

$$\| E(m, m', ib_{n-1} + s, ib') \| \leqq K_a(1 + | m | + | m' | + | b_{n-1} + b | + | b' |)^3$$

where $s = a + ib$ and $-1 < a < 1$. It now follows easily from (9.8) that $L(m, m', ib_{n-1} + s, ib')$ can be continued so as to be analytic in $-1 < Re(s) < 1$ with the same (operator) bound as $E(m, m', ib_{n-1} + s, ib')$. In view of (9.7) and the fact that $T(p, \epsilon)$ is unitary the same can be said of $Q_1(\lambda, s)$. To obtain the bound given in (2), we need only observe that $| m | + | m' | \leqq 2(\sum | m_j |)$ and that $| b' | \leqq \sum | b_j |$. This concludes the proof of the lemma.

Since we frequently need to refer to the following elementary result, we shall state it as a lemma.

LEMMA 25. *Let F_1 and F_2 be analytic complex or operator valued functions defined on Λ^*. Then $F_1 = F_2$ if $F_1(\lambda) = F_2(\lambda)$ for all $\lambda \in \Lambda$.*

LEMMA 26. *Let $H(\lambda) = H(m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$ be defined for $\lambda \in \Lambda^*$ by*

$$(9.9) \qquad H(\lambda) = (1 + \sum_j | m_j |)^3 \prod_j (3 - s_j)^6.$$

If $0 < d < 1$ let $\Lambda^*(d)$ denote the set of all $\lambda \in \Lambda^*$ such that $-d < Re(s_j) < d$, $j = 1, 2, \cdots, n$. Then the operator valued function $R(p, \lambda)$ initially defined for $\lambda \in \Lambda$ can be continued into Λ^* in such a way that

(1) *for each f, g in $L_2(V)$, the function*

$$\lambda \to (R(p, \lambda)f, g)$$

is analytic in Λ^*, and

$$(2) \qquad \| R(p, \lambda) \| \leqq K_d' \, | H(\lambda) |, \quad \lambda \in \Lambda^*(d)$$

where K_d' is independent of λ.

Proof. Let $\lambda = (m_1, \cdots, m_n; ib_1, \cdots, ib_n) \in \Lambda$. Then if $-1 < Re(s) < 1$ we shall put

$$\lambda_s = (m_1, \cdots, m_n; ib_1, \cdots, ib_{n-1} + s, ib_n - s).$$

It follows that $\lambda_s \in \Lambda^*$. Now in view of (9.2) and Lemma 24 we may extend the domain of definition of $R(p, \lambda)$ by setting

$$R(p, \lambda_s) = Q(\lambda, s), \quad -1 < Re(s) < 1. \tag{9.10}$$

Next we define our operator for permutations of λ_s. To be specific, let us suppose $k_1, k_2, \cdots, k_n$ is a permutation of $1, 2, \cdots, n$. Then there is an element q of the Weyl group, i. e., an element q of S such that

$$q\lambda = (m_{k_1}, \cdots, m_{k_n}; ib_{k_1}, \cdots, ib_{k_n}).$$

By Theorem 3, $R(p, \lambda) = R(p, q\lambda)$ for all λ in Λ. We may therefore obtain an extension of our operator, which is invariant under S, by setting

$$R(p, q\cdot\lambda_s) = R(p, \lambda_s), \quad -1 < Re(s) < 1. \tag{9.11}$$

From the bound given in Lemma 24 it is easy to see that $\| Q(\lambda, s) \| \leqq K_a \,|\, H(\lambda) \,|$. Thus since H is invariant under permutations it follows at once from (9.10) that

$$\| R(p, q\cdot\lambda_s) \| \leqq K_a \,|\, H(q\cdot\lambda_s) \,|$$

To simplify the notation, let Λ' denote the set of all characters of the form $q\cdot\lambda_s$, $\lambda \in \Lambda$, $q \in S$. *Then* Λ' can also be described as the set of all characters in Λ^* of the form

$$\lambda = (m_1, \cdots, m_n; ib_1, \cdots, ib_j + s, \cdots, ib_k - s, \cdots, ib_n) \tag{9.12}$$

where $1 \leqq j < k \leqq n$ and $-1 < Re(s) < 1$. At this point we have succeeded in defining $R(p, \lambda)$ for all characters λ in Λ'. Furthermore we have the bound

$$\| R(p, \lambda) \| \leqq K_a \,|\, H(\lambda)) \tag{9.13}$$

where λ has the form (9.12) and $s = a + ib$. If f and g belong to $L_2(V)$ and $\lambda \in \Lambda'$ we set

$$\Phi(f, g, \lambda) = (R(p, \lambda) f, g). \tag{9.14}$$

If $0 < d < 1$ and $K_d' = \sup_{-d<a<d} K_a$ it follows from (9.13) that

$$|\Phi(f,g,\lambda)| \leqq K_d' \,|H(\lambda)|\, \|f\|_2 \|g\|_2 \tag{9.15}$$

for all $\lambda \in \Lambda' \cap \Lambda^*(d)$, i. e., all λ if the form (9.12) where $-d < Re(s) < d$. Next we fix f and g and show that $\Phi(f,g,\lambda)$, as a function of λ, can be extended analytically into $\Lambda^*(d)$. In doing this, we may of course assume that $\|f\|_2 \|g\|_2 \neq 0$. We put

$$F(\lambda) = \frac{\Phi(f,g,\lambda)}{K_d' \cdot H(\lambda) \cdot \|f\|_2 \|g\|_2}, \quad \lambda \in \Lambda' \cap \Lambda^*(d).$$

Then for fixed $m_1, \cdots, m_n$ and fixed $b_1, b_2, \cdots, b_n$ the function

$$F_{jk}(s) = F(m_1, \cdots, m_n; ib_1, \cdots, ib_j + s, \cdots, ib_k - s, \cdots, ib_n)$$

is analytic in $-d < Re(s) < d$. This follows from Lemma 24, (9.10), (9.11), (9.14), and the fact that $1/H$ is analytic in Λ^*. Furthermore F_{jk} is bounded by 1. It now follows from Lemma 24 that F has an analytic extension into $\Lambda^*(d)$ which is also bounded by 1. Thus $\Phi(f,g,\lambda)$ has an analytic extension to $\Lambda^*(d)$ which satisfies (9.15). Moreover, since

$$\Lambda^* = \bigcup_{0<d<1} \Lambda^*(d)$$

our argument shows that $\Phi(f,g,\lambda)$ can be extended into Λ^*. If $\lambda \in \Lambda$ is fixed, it follows from (9.14) that $f, g \to \Phi(f,g,\lambda)$ is a sesqui-linear form on $L_2(V)$. By Lemma 25 the same is true for any fixed $\lambda \in \Lambda^*$. In addition

$$|\Phi(f,g,\lambda)| \leqq K_d' \,|H(\lambda)|\, \|f\|_2 \|g\|_2$$

for all λ in $\Lambda^*(d)$. Therefore, $\Phi(\cdot,\cdot,\lambda)$ defines for each $\lambda \in \Lambda^*$, a bounded operator which we denote by $R(p,\lambda)$. It is then clear that $R(p,\lambda)$ satisfies statements (1) and (2) of the lemma.

Proof of Theorem 4. (1) Let $\lambda = (m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n) \in \Lambda^*$, and $r = \sum m_j$. Then, extending the definition given in Section 6, we shall put

$$\operatorname{res} \lambda = (0, \cdots, r; 0, \cdots, 0)$$

and call $\operatorname{res}\lambda$ the residue of λ. Suppose now that $a \in G_0$, the sub-group of all elements a such that $a_{jn} = 0$, $1 \leqq j \leqq n-1$. Then in view of Theorem 3, we may define $R(a,\lambda)$ in a consistent, in fact analytic, fashion by setting

$$R(a,\lambda) = R(a, \operatorname{res}\lambda). \tag{9.16}$$

On the other hand if $a \in G$ and is not a member of G_0, then, by Lemma 14, $a \in G_0 p G_0$ where $p = p_{n-1}$. We can then write $a = bpb'$ with b and b' in G_0. By (9.16) and Lemma 26, the function

$$\lambda \to R(b,\lambda)R(p,\lambda)R(b',\lambda)$$

is analytic in Λ^*. In addition

$$R(a,\lambda) = R(b,\lambda)R(p,\lambda)R(b',\lambda), \quad \lambda \in \Lambda. \tag{9.17}$$

It follows from Lemma 25 and (9.17) that the function

$$\lambda \to R(b,\lambda)R(p,\lambda)R(b',\lambda)$$

depends only on a and not on the particular choice of b and b'. It is therefore natural to extend the definition of $R(a,\lambda)$ by setting

$$R(a,\lambda) = R(b,\lambda)R(p,\lambda)R(b',\lambda)$$

for each $\lambda \in \Lambda^*$.

(2) Let a and b belong to G. Then by (1), $R(a,\lambda)$, $R(b,\lambda)$, and $R(ab,\lambda)$ are all analytic, as functions of λ, in Λ^*. Since the product of two analytic operator valued functions in again analytic, $R(a,\lambda)R(b,\lambda)$ is analytic in Λ^*. Furthermore,

$$R(a,\lambda)R(b,\lambda) = R(ab,\lambda) \tag{9.18}$$

whenever $\lambda \in \Lambda$. Applying Lemma 25 we see that (9.18) holds for all $\lambda \in \Lambda^*$. This shows $a \to R(a,\lambda)$ is a representation.

Next we show that when λ is fixed the operators $R(a,\lambda)$ are uniformly bounded as functions of $a \in G$. If $a \in G_0$, then $R(a,\lambda)$ is unitary and hence $\| R(a,\lambda) \| = 1$. On the other hand, if $a \in G$ and is not in G_0, $a = bpb'$, as above, with b, b' in G_0. It follows that $\| R(a,\lambda) \| = \| R(p,\lambda) \|$, and for the bound of $R(p,\lambda)$ we have the estimate given by Lemma 26.

Now suppose f, g are fixed functions in $L_2(V)$ and that $\{a_k\}$ is a sequence of elements in G tending to the identity. Let us consider the sequence of functions

$$F_k(\lambda) = (R(a_k,\lambda)f, g), \quad \lambda \in \Lambda^*.$$

We shall show that for each fixed λ, there is a subsequence F_{k_i} such that $F_{k_i}(\lambda) \to (f,g)$. From this it is easy to conclude that for each fixed λ,

$a \to (R(a,\lambda)f,g)$ is continuous, and this being true for each f, g it will follow that $a \to R(a,\lambda)$ is (weakly) continuous. If

$$\lambda = (m_1, m_2, \cdots, m_m; s_1, s_2, \cdots, s_n) \in \Lambda^*$$

then we can find d, d' so that

$$(9.19) \qquad -d < Re(s_j) < \mathrm{d}, \ -d' < I_m(s_j) < d', \ 1 \leqq j \leqq n.$$

Let $\Lambda(m_1, m_2, \cdots, m_m; d, d')$ be the set of all characters in Λ^* of the form

$$\lambda = (m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$$

which satisfy (9.19). Now fix $m_1, m_2, \cdots, m_n$, d and d' and set $\Lambda' = \Lambda(m_1, m_2, \cdots, m_n; d, d')$. Then from the bound given in Lemma 27, it follows that there is a constant K independent of k such that

$$(9.20) \qquad |F_k(\lambda)| \leqq K. \ \lambda \in \Lambda'.$$

Since the functions F_k are analytic in Λ' and satisfy (9.20) they are equicontinuous there. Hence there is a subsequence F_{k_i} which converges uniformly on Λ' to a function F which is also analytic in Λ'. We shall show that $F(\lambda) = (f,g)$ for all $\lambda \in \Lambda'$. To do this we make use of the known fact that the representations of the principal series are continuous. From this and (6.6) we see that $a \to R(a,\lambda)$ is continuous whenever λ is a unitary character, i.e., when $\lambda \in \Lambda$. Hence $F_k(\lambda) \to (f,g)$, if $\lambda \in \Lambda$. In particular, this implies $F(\lambda) = (f,g)$ for all $\lambda \in \Lambda \cap \Lambda'$. Thus by analytic continuation we see that $F(\lambda) = (f,g)$ for all $\lambda \in \Lambda'$.

(3) Let a be fixed in G and q an element of S. Then $R(a, q\lambda)$ is analytic as a function of λ and so is $R(a,\lambda)$; in addition(these functions agree when $\lambda \in \Lambda$. By Lemma 25, $R(a,q\lambda) = R(a,\lambda)$ for all $\lambda \in \Lambda^*$.

(4) Suppose a is a fixed element of G. Then to prove $R(a,\lambda') = R(a,\lambda)'$ we must show that $R(a,\lambda') = R(a^{-1},\lambda)^*$. Let $f, g \in L_2(V)$, and put $F(\lambda) = (R(a,\lambda)g,f)$. Then F is analytic in Λ^*, and if $\lambda = (m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$

$$\bar{F}(\lambda'))= \bar{F}(m_1, m_2, \cdots, m_n; -\bar{s}_1, -\bar{s}_2, \cdots, -\bar{s}_n).$$

It follows easily that $\bar{F}(\lambda')$ is analytic, as a function of λ, in Λ^*. In addition

$$(9.21) \qquad \bar{F}(\lambda') = (f, R(a,\lambda')g), \ \lambda \in \Lambda^*.$$

On the other hand, if λ is unitary then $\lambda=\lambda'$ and $R(a,\lambda)=R(a^{-1},\lambda)^*$. Thus we have the relation

$$(R(a^{-1},\lambda)f,g)=(f,R(a,\lambda')g),\ \lambda\in\Lambda. \tag{9.22}$$

Since both sides of (9.22) are analytic in Λ^* (as functions of λ), it follows from Lemma 25 that (9.22) holds for all $\lambda\in\Lambda^*$. This being true for every f and g in $L_2(V)$. we conclude that $R(a^{-1},\lambda)^*=R(a,\lambda')$, $\lambda\in\Lambda^*$.

10. Traces and equivalences. In the first theorem of this section we obtain a generalization of a trace formula due to Gelfand and Neumark, [3: p. 93]. With the aid of this formula we then determine the conditions under which two representations of our family

$$a\to R(a,\lambda),\ \lambda\in\Lambda^*$$

are equivalent. In particular we show that not all of our representations are equivalent to unitary representations.

We begin by recalling some rather standard facts. If $\mathfrak{H}$ is a Hilbert space and A is a bounded operator on $\mathfrak{H}$, the L_p norm of A is defined by the equation

$$\|A\|_p=[\mathrm{trace}(A^*A)^{p/2}]^{1/p},\ 1\leqq p<\infty \tag{10.1}$$

and $\|A\|_\infty$ *is* the usual operator norm, $\|A\|$. If $\|A\|_1<\infty$ then $\mathrm{tr}(A)=\mathrm{trace}\,(A)$ exists, $|\mathrm{tr}(A)|\leqq\|A\|_1$, and A is said to be of trace class. On the other hand, if A and B are any two bounded operators on $\mathfrak{H}$ then

$$\|AB\|_p\leqq\|A\|_\infty\|B\|_p,\ 1\leqq p\leqq\infty. \tag{10.2}$$

Next, let $\lambda\in\Lambda^*$ and $a\in G$. If a has distinct characteristic values $c_1,c_2,\cdots,c_n$ let $c=[c_1,c_2,\cdots,c_n]$ and

$$\psi(a,\lambda)=\sum_p\frac{\lambda(p^{-1}cp)}{D(a)} \tag{10.3}$$

where $D(a)=\prod_{j<k}|c_j-c_k|^2$ and p ranges over S. If a does not have distinct characteristic values set $\psi(a,\lambda)=0$. Then $\psi(a,\lambda)$ is a well defined measurable function of a.

THEOREM 5. *Let f be the convolution of two bounded functions with compact support on G. For each $\lambda \in \Lambda^*$, let*

$$R(f,\lambda) = \int_G R(a,\lambda) f(a)\,da.$$

Then $R(f,\lambda)$ is of trace class and

$$\operatorname{tr} R(f,\lambda) = \int_G \psi(a,\lambda) f(a)\,da \tag{10.4}$$

where $\psi(a,\lambda)$ is defined by (10.3).

Equation (10.4) is a generalization of the formula of Gelfand and Neumark [3: p. 93].

THEOREM 6. *Let λ_1 and λ_2 belong to Λ^*, and suppose A is a bounded operator on $L_2(V)$ with a bounded inverse such that*

$$R(a,\lambda_1) = AR(a,\lambda_2)A^{-1}, \quad a \in G.$$

Then there is an element q in the Weyl group, S such that $\lambda_1 = q\lambda_2$, and then, by Theorem 4, $R(a,\lambda_1) = R(a,\lambda_2)$, $a \in G$.

On the other hand, suppose $\lambda \in \Lambda^*$ and that

$$a \to AR(a,\lambda)A^{-1}$$

is a unitary representation, where again A is a bounded operator on $L_2(V)$ with a bounded inverse. Then $a \to R(a,\lambda)$ is already a unitary representation, and there is an element q in S such that $\lambda' = q\lambda$. Thus if $\lambda' \neq q\lambda$ for all $q \in S$, $a \to R(a,\lambda)$ is not equivalent to a unitary representation.

For the proofs of these theorems we require a series of lemmas.

LEMMA 27. *Let U be a compact group and $u \to R(u)$ a continuous representation of U on a Hilbert space $\mathfrak{H}$. Suppose*

$$\sup_{u \in U} \| R(u) \| \leqq M.$$

Then there exists a bounded invertible operator B on $\mathfrak{H}$ such that

(1) $\| B \| \leqq M$, $\| B^{-1} \| \leqq M$, *and*

(2) $u \to BR(u)B^{-1}$ *is a unitary representation of U.*

Proof . Let $x \in \mathfrak{H}$ and suppose $\| x \| = 1$. Then for $u \in U$,

$$\frac{1}{M^2} \leqq (R^*(u)R(u)x, x) \leqq M^2.$$

Hence if $A = \int_U R^*(u)R(u)\,du$ and the measure of U is properly normalized

$$\frac{1}{M^2} \leqq (Ax, x) \leqq M^2.$$

If B is the positive square root of A then $\| B \| \leqq M$ and $\| B^{-1} \| \leqq M$ as well. Part (2) now follows by the usual argument [2].

LEMMA 28. *Let f be a bounded function with compact support on G*

$$R(f, \lambda) = \int_G f(a)R(a, \lambda)\,da, \quad \lambda \in \Lambda^*$$

and

$$K_\lambda = \sup_{a \in G} \| R(a, \lambda) \|.$$

Then $\| R(f, \lambda) \|_2 < K_\lambda{}^5 J_f$ *where* J_f *is a constant depending only on* f *and not on* λ.

This is the key lemma.[9]

Proof. Recall that $R(a, \lambda) \,|_{G_0}$ is unitary and irreducible.[10] Hence $R(a, \lambda)$ is completely irreducible in the terminology of Godement [5]. Let U be the unitary sub-group of G, and $u \to D(u)$ a continuous irreducible representation of U. Let $n(D)$ be the number of times that D occurs in the reduction of $R(a, \lambda) \,|_U$. Then since $R(a, \lambda)$ is completely irreducible it follows from a theorem of Godement [5: Theorem 2] that $n(D) \leqq \text{degree}\,(D)$. Moreover, by Lemma 27, there exists a bounded operator B_λ such that $\| B_\lambda \| \leqq K_\lambda$, $\| B_\lambda{}^{-1} \| \leqq K_\lambda$, and

$$P(a, \lambda) = B_\lambda R(a, \lambda) B_\lambda{}^{-1}$$

is unitary when restricted to U. Since $\| P(a, \lambda) \| \leqq \| B_\lambda \| \; \| R(a, \lambda) \| \; \| B_\lambda{}^{-1} \|$, it follows that

$$\sup_{a \in G} \| P(a, \lambda) \| \leqq K_\lambda{}^3,$$

[9] The proof makes use of ideas occurring in paper [6] of Harish-Chandra.

[10] See (9.16), Theorem 3 and subsequent remarks. This is the first place where we use the irreducibility of the principal series on G_0.

Because $P(a,\lambda)$ is equivalent to $R(a,\lambda)$, $n(D)$ is also the number of times D occurs in the reduction of $P(a,\lambda)\,|_U$.

Now let S be the closure of the support of f. Then S is compact and so is $U\cdot S$. Fixing λ and setting $P(a)=P(a,\lambda)$ we then have

$$P_f=\int_G f(a)P(a)\,da=\int_U du\int_G f(ua)P(ua)\,da.$$

From this it follows that

$$\|P_f\|_2=\|\int_G da\int_U f(ua)P(ua)\,du\,\|_2\leqq\int_G\,\|\int_U f(ua)P(ua)\,du\,\|_2\,da.$$

Hence, since $\int_U f(ua)P(ua)\,du=(\int_U f(ua)P(u)\,du)P(a)$ it follows from (10.2) that

$$\|P_f\|_2\leqq\sup_{a\in G}\|P(a)\|_\infty\int_G\,\|\int_U f(ua)P(u)\,du\,\|_2\,da.$$

We now estimate $\|\int_U f(ua)P(u)\,du\,\|_2$. Let $u\rightarrow L(u)$ be the left regular representation of U. Then D occurs exactly degree (D) times in L. Since $n(D)\leqq$ degree (D) it follows that

$$\|\int_U f(ua)P(u)\,du\,\|_2\leqq\|\int_U f(ua)L(u)\,du\,\|.$$

In addition, the Peter-Weyl theorem shows that

$$\|\int_U f(ua)L(u)\,du\,\|_2=(\int_U|f(ua)|^2\,du)^{\frac{1}{2}}\leqq\sup_{b\in G}|f(b)|.$$

Hence because f has its support in S

$$\begin{aligned}\int_G\,\|\int_U f(ua)P(u)\,du\,\|_2\,da&=\int_{U\cdot S}\|\int_U f(ua)P(u)\,du\,\|_2\,da\\&\leqq\|f\|_\infty\,\text{meas}\,(U\cdot S)\\&=J_f.\end{aligned}$$

Combining these estimates we find that

$$\| P_f \|_2 \leqq K_\lambda^3 J_f.$$

However, $R(f,\lambda) = B_\lambda^{-1} P_\lambda B_\lambda$. Thus

$$\| R(f,\lambda) \|_2 \leqq \| B_\lambda^{-1} \|_\infty \| P_f \|_2 \| B_\lambda \|_\infty \leqq K_\lambda^5 J_f$$

which proves the lemma.

LEMMA 29. *Let $f = f_1 * f_2$ where f_1 and f_2 are bounded functions with compact support on G. Then for each $\lambda \in \Lambda^*$*

$$R(f,\lambda) = \int_G f(a) R(a,\lambda)\, da$$

is of trace class, $\| R(f,\lambda) \|_1 \leqq K_\lambda^{10} J_f'$, and

$$\lambda \to \operatorname{tr} R(f,\lambda)$$

is analytic in Λ^*.

Proof. Since $f = f_1 * f_2$, $R(f,\lambda) = R(f_1,\lambda) R(f_2,\lambda)$. Hence $\| R(f,\lambda) \|_1 \leqq \| R(f_1,\lambda) \|_2 \| R(f_2,\lambda) \|_2$, by Schwarz's inequality, which is valid in the present situation. The preceding lemma now shows that

$$\| R(f,\lambda) \|_1 \leqq K_\lambda^{10} J_{f_1} J_{f_2} = K_\lambda^{10} J_f'.$$

Thus $\operatorname{tr} R(f,\lambda)$ is well defined.

Let $g_1, g_2, \cdots$, be a fixed orthonormal basis for $L_2(V)$, and let P_n be the orthogonal projection on the subspace spanned by $g_1, g_2, \cdots, g_n$. Then if A is any operator on $L_2(V)$ of trace class we know that

$$\operatorname{tr}(A) = \lim_{n\to\infty} \operatorname{tr}(P_n A P_n)$$

and $| \operatorname{tr}(P_n A P_n) | \leqq \| A \|_1$. Furthermore

$$\operatorname{tr}(P_n A P_n) = \sum_{k=1}^{n} (A g_k, g_k). \tag{10.5}$$

Now let $F_n(\lambda) = \operatorname{tr}(P_n R(f,\lambda) P_n)$ and $F(\lambda) = \operatorname{tr}(R(f,\lambda))$. In view of

(10.5) and the fact that $\lambda \to R(a,\lambda)$ is analytic in Λ^*, and hence that $\lambda \to R(f,\lambda)$ is analytic there, it follows that F_n is analytic in Λ^*. We also have the estimate

$$|F_n(\lambda)| \leqq \| R(f,\lambda)\|_1 \leqq K_\lambda{}^{10} J_f'.$$

Since K_λ is uniformly bounded over every compact subset of Λ^*, the same is true of the sequence $F_n(\lambda)$. As $F_n(\lambda) \to F(\lambda)$ for every $\lambda \in \Lambda^*$, it therefore follows that F is analytic in Λ^*.

LEMMA 30. *Let f be a bounded function with compact support on G and $\psi(a,\lambda)$ the function defined by* (10.3). *Then the function*

$$H(\lambda) = \int_G f(a)\psi(a,\lambda)\,da \tag{10.6}$$

is analytic in Λ^*.

Proof. Recall that if $\lambda = (m_1, m_2, \cdots, m_n; s_1, s_2, \cdots, s_n)$ then

$$\lambda(c) = \prod_{j=1}^{n} \left(\frac{c_j}{|c_j|}\right)^{m_j} |c_j|^{s_j}.$$

Furthermore

$$H(\lambda) = \sum_p \int_G \lambda(p^{-1}cp)\frac{f(a)}{D(a)}\,da$$

and $\dfrac{f(a)}{D(a)}$ is integrable and has compact support. The analyticity of H, i.e., the existence of the complex derivatives $\dfrac{\partial H}{\partial s_1}, \cdots, \dfrac{\partial H}{\partial s_{n-1}}$, follows by straightforward differentiation under the integral sign, these operations being justified by the absolute convergence of the integrals in question.

Proof of Theorem 5. If λ is a unitary character, then by its definition $a \to R(a,\lambda)$ is unitarily equivalent to the corresponding representation $a \to T(a,\lambda)$ of the principal series. Thus if f is the convolution of two bounded functions with compact support on G then

$$\operatorname{tr} R(f,\lambda) = \operatorname{tr} T(f,\lambda), \quad \lambda \in \Lambda.$$

On the other hand, Gelfand and Neumark have shown that

$$\operatorname{tr} T(f,\lambda) = \int_G f(a)\psi(a,\lambda)\,da = H(\lambda).$$

But if $F(\lambda) = \operatorname{tr} R(f,\lambda)$ then F is analytic in Λ^*, by Lemma 29 while H is analytic in Λ^*, by Lemma 30. Since $F(\lambda) = H(\lambda)$ for $\lambda \in \Lambda$, i.e., for unitary characters, it follows from Lemma 25 that $F(\lambda) = H(\lambda)$ for all λ in Λ^*. This finishes the proof of Theorem 5.

Proof of Theorem 6. Suppose $R(a,\lambda_1) = AR(a,\lambda_2)A^{-1}$ where A is a bounded operator with a bounded inverse. Then if

$$R(f,\lambda) = \int_G f(a)R(a,\lambda)\,da, \quad \lambda \in \Lambda^*$$

it follows that, $R(f,\lambda_1) = AR(f,\lambda_2)A^{-1}$. Now let f be the convolution of two bounded functions with compact support. Then $R(f,\lambda_1)$ and $R(f,\lambda_2)$ are both of trace class and by a standard property of the trace we see that

$$\operatorname{tr}(R(f,\lambda_1)) = \operatorname{tr}(R(f,\lambda_2)).$$

Thus by Theorem 5

$$\int_G [\psi(a,\lambda_1) - \psi(a,\lambda_2)]f(a)\,da = 0. \tag{10.7}$$

By a simple limiting argument (10.7) also holds whenever f is bounded and has bounded support. Hence $\psi(a,\lambda_1) = \psi(a,\lambda_2)$ a.e.. From this, (10.3) and the continuity of λ_1 and λ_2 we conclude that

$$\sum_p (p\lambda_1)(c) = \sum_p (p\lambda_2)(c)$$

for all diagonal matrices c in G. But since distinct characters are linearly independent, it follows that $\lambda_1 = q\lambda_2$ for some $q \in S$.

Now suppose $a \to T(a)$ is a unitary representation of G and that $T(a) = AR(a,\lambda)A^{-1}$. Then since $T^*(a) = T^{-1}(a) = T(a^{-1})$, we see that

$$A^{*-1}R(a,\lambda)^*A^* = AR(a^{-1},\lambda)A^{-1}$$

or in other words, that

$$R(a,\lambda)^*A^*A = A^*AR(a^{-1},\lambda).$$

But when $a \in G_0$, $R(a,\lambda)$ is unitary and irreducible. This implies $A^*A = cI$, where $c > 0$. After proper normalization we may therefore assume A is a unitary operator. Hence $a \to R(a,\lambda)$ is a unitary representation. It follows from Theorem 4 that $R(a,\lambda) = R(a,\lambda')$ where λ' is the contragredient of λ. Thus by the first statement of the present theorem, $\lambda' = q\lambda$ for some q in S.

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IMAGES OF E_n UNDER ACYCLIC MAPS.*

By E. H. Connell.[1]

It has been shown in [8] that monotone mappings of E_2 onto E_2 are compact. In this paper it will be shown that acyclic mappings of E_n onto E_n are compact. It is an open question whether or not monotone mappings of E_3 on E_3 must be compact. It will then be shown that contractible open 3-manifolds embedded in E_3, but topologically distinct from E_3, cannot be images of E_3 under a class of acyclic maps. It is unknown whether or not any contractible 3-manifold distinct from E_3 can be such an image.

Notation. A map (continuous function) is monotone if each point-inverse is a compact connected set. A map is compact if the inverse of each compact set is compact. A map is acyclic if each point-inverse is a compact connected set with vanishing n-th homology groups for $n = 1, 2, 3 \cdots$ (Cech homology over a field G). If X is a space, $(X + \infty)$ is the one point compactification of X. If $B \subset X$, $C(B)$ is the complement of B. If $f\colon X \to Y$ and $p \in Y$, then $f^{-1}(p) \subset X$ will be called an orbit, and if $A \subset X$, the saturation of A is the union of all orbits intersecting A, i.e. saturation of $A = f^{-1}(f(A))$. Manifolds are without boundary, i.e. a manifold is a connected separable metric space which is locally Euclidean.

Theorem 1. *If f is an acyclic map of $E_n (n > 1)$ onto Y, a contractible n-manifold, then f is compact.*

Proof. Suppose for each $p \in Y$, $\exists$ a neighborhood N of p such that $f^{-1}(N)$ is bounded. Then the inverse of each compact set would be closed and bounded, and the theorem would be trivial. Thus it may be supposed that $\exists p \in Y$ such that if O is an open set containing p, $f^{-1}(O)$ is unbounded. Let $M = f^{-1}(p)$ and let S_1 be an $(n-1)$-sphere such that M is in the bounded complementary domain of S_1. Let S_2 be the saturation of S_1 and note that S_2, like S_1, is compact, connected, does not intersect M and separates M from ∞. (To see that S_2 is compact, first show it is bounded. Suppose p_n is a sequence in S_2 which approaches ∞, and A_n is the sequence of orbits with $p_n \in A_n$. Let $q_n \in A_n \cap S_1$ and note that some subsequence, also denoted

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by q_n, converges to $q \in S_1$. Let A be the orbit containing q and V be a bounded open set containing A. It is easy to show, using the fact that each A_n is connected, that all but finitely many of the A_ns are contained in V. This is a contradiction and shows that S_2 is bounded. A similar argument shows that S_2 is closed and thus compact.) Let O_1 be the bounded complementary domain of S_2 which contains M and O_2 be the unbounded complementary domain of S_2. Let $S = E_n - C(O_1 \cup O_2)$. Note that S is merely S_2 plus all bounded complementary domains of S_2 which do not contain M. Thus S is compact and connected and since it separates E_n into exactly two pieces, $H_{n-1}(S) = G$. By the Vietoris mapping theorem ([6] and [1]), $H_{n-1}(f(S)) = G$. Since Y is a contractible n-manifold, $f(S)$ separates Y into two pieces, one unbounded (i.e. noncompact closure) and one bounded. Since f is onto, $f(O_2)$ is the unbounded complementary domain and $f(O_1)$ the bounded one. Now the bounded one is an open set containing p whose inverse is the bounded set O_1. This is a contradiction and proves the theorem.

It should be noted that Whyburn's theorem about E_2 is not obtained as a corollary, since he assumes only that inverses are compact and connected. However we do obtain the following:

COROLLARY 1. *If f is an acyclic map of E_n onto Y, a contractible n-manifold, and f is extended to $f\colon (E_n + \infty) \to (Y + \infty)$ by $f(\infty) = \infty$, then f as extended is continuous.*

Proof. Let $x_n \to \infty$. *If $f(x_n) \not\to \infty$ then $\exists$ a subsequence, also denoted* x_n, such that $f(x_n) \to y \in Y$. Let $N \subset Y$ be a compact neighborhood of y; then by Theorem 1, $f^{-1}(N)$ is compact, but this is impossible.

COROLLARY 2. *If f is an acyclic map of E_n onto a contractible n-manifold Y, the natural map from the decomposition space generated by f onto Y is a homeomorphism.*

Proof. By Corollary 1, f may be extended to the one point compactification. For compact spaces, this result is well known (p. 126 of [9]) and the desired result is obtained by removing the point at infinity.

Note that each of the above corollaries implies, and each is thus equivalent to, Theorem 1. The following corollary is well known.

COROLLARY 3. *If f is a one-to-one mapping of E_n onto E_n, then f is a homeomorphism.*

Proof. The decomposition space is simply E_n itself, and the result follows directly from Corollary 2.

R. H. Bing has posed the question: if a point-like decomposition of E_3 yields a 3-manifold, must it be E_3? Another question would be: if Y is a 3-manifold and f maps E_3 onto Y such that point inverses are point-like, then must Y be E_3? If a set is point-like, it is cellular (see [2]) and by the continuity of Cech homology, must have the homology of a cell, i.e., must be acyclic. Thus by Corollary 2, these two questions are the same. At any rate, the question is partially answered here. The author thanks R. McMillan for his assistance.

THEOREM 2. *Suppose that Y is a contractible 3-manifold with a locally-finite triangulation and that any finite subcomplex of Y can be semilinearly embedded in E_3. Suppose f maps E_3 onto Y with each $f^{-1}(p)$ a compact absolute retract (i.e. contractible and locally contractible). Then Y is homeomorphic to E_3.*

Proof. If Y is the monotonic union of closed 3-cells, each containing the previous in its interior, then Y is homeomorphic to E_3. Thus it will suffice to show that for each compact subset N of Y, $\exists$ a 3-cell containing N.

Suppose $N \subset Y$ is compact. By Theorem 1, $f^{-1}(N)$ is compact. Let M be the saturation of a closed 3-cell containing $f^{-1}(N)$. Let 0 be the unbounded complementary domain of M and note that $\pi_2(0) \neq 0$. Since f is compact, Smale's theorem (see [5]) applies and $\pi_2(f(0)) \neq 0$. Now by the sphere theorem ([4] and [7]), there exists a polyhedral 2-sphere S in $f(0)$ which cannot be shrunk to a point in $f(0)$. It plus its bounded complementary domain $I(S)$ can be embedded in E_3 by hypothesis, and by Alexander's theorem, S must bound a 3-cell; thus $S \cup I(S)$ is a 3-cell. Since S can be shrunk in $S \cup I(S)$, $I(S) \cap f(M) \neq \phi$, and since M is connected, $f(M) \subset I(S)$ and therefore $N \subset I(S)$. Q. E. D.

The Poincaré conjecture is equivalent to the following: if Y is a contractible 3-manifold, then each compact subset of Y can be embedded in E_3. This is contained implicitly in [3]. R. McMillan and J. Kister have constructed a contractible 3-manifold such that each compact subset can be embedded in E_3, but the manifold itself cannot be (yet unpublished).

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INDEX

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[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Société Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

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